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ESCAPE OF MASS AND ENTROPY FOR DIAGONAL FLOWS IN REAL RANK ONE SITUATIONS

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ABSTRACT. Let G be any connected semisimple Lie group of real rank 1 with finite center, let Γ be any non-uniform lattice in G and a any diagonalizable element in G. We investigate the relation between the metric entropy of a acting on the homogeneous space $\Gamma\backslash G$ and escape of mass. Moreover, we provide bounds on the escaping mass and, as an application, we show that the Hausdorff dimension of the set of orbits (under iteration of a) which miss a fixed open set is not full.

CONTENTS

1. INTRODUCTION

Let G be a connected semisimple Lie group of $\mathbb{R}$-rank 1 with finite center and Γ a lattice in G. Suppose that

$$\mathcal{X} := \Gamma\backslash G$$

denotes the arising homogeneous space. Let A be a maximal one-parameter subgroup consisting of diagonalizable elements. Pick an element $\tilde{a} \in A \setminus \{\mathrm{id}\}$

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and consider the right action

$$T\colon \begin{cases} \mathcal{X} & \to & \mathcal{X} \\ x & \mapsto & x\tilde{a} \end{cases}$$

of $\tilde{a}$ on $\mathcal{X}$. Further let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of T-invariant probability measures on $\mathcal{X}$ which converges in the weak* topology to the measure ν.

If ν is itself a probability measure (which is always the case if Γ is cocompact), then upper semi-continuity of metric entropy is well-known, that is

$$\limsup_{n\to\infty} h_{\mu_n}(T) \le h_\nu(T).$$

In this article we investigate the case that Γ is non-cocompact and ν is not a probability measure. We show that if upper semi-continuity does not hold, the amount by which it fails is controlled by the escaping mass. More precisely, the main result can be stated as follows.

Theorem. *Let $h_m(T)$ denote the maximal entropy of T and suppose that $\nu(\mathcal{X}) > 0$. Then*

$$\nu(\mathcal{X}) h_{\frac{\nu}{\nu(\mathcal{X})}}(T) + \tfrac{1}{2} h_m(T)\cdot(1-\nu(\mathcal{X})) \ge \limsup_{n\to\infty} h_{\mu_n}(T).$$

A consequence of this theorem is the following result about escape of mass, which is of interest on its own.

Corollary. *Suppose that $\limsup h_{\mu_n}(T) \ge c$. Then*

$$\nu(\mathcal{X}) \ge \frac{2c}{h_m(T)} - 1.$$

Thus, if the entropy on the sequence (μ_n) is high, meaning at least $\frac{1}{2}h_m(T)+\varepsilon$, then not all of the mass can escape and the remaining mass can be bounded quantitively.

For $\mathcal{X} = \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$ and T being the time-one map this control on escape of mass is already shown in [ELMV]. Moreover, [EK] provides a result of this kind for the rank 2 space $\mathcal{X} = \mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R})$ and the action by a specific singular diagonal element. We refer to [EK] for a discussion of applications.

In the extreme case of escape of mass, the theorem and corollary above yield the following consequence for the entropy of the remaining normalized measure.

Corollary. *If $\limsup h_{\mu_n}(T) \ge c$ and*

$$\nu(\mathcal{X}) = \frac{2c}{h_m(T)} - 1 > 0,$$

then $h_{\frac{\nu}{\nu(\mathcal{X})}}(T)$ is the maximal entropy $h_m(T)$.

As an application of these results and the methods for their proofs we show in Section 8 that the Hausdorff dimension of the set of forward-A-orbits which miss a fixed open set is not full, thereby answering a question of Barak Weiss. Its positive solution is already used in [HW] and [KW].

Let us outline the strategy of proof for the main theorem. The key tool for its proof is the existence of a finite partition η of $\mathcal{X}$ such that for each T-invariant probability measure μ on $\mathcal{X}$ the entropy of μ, the entropy of the partition η and

the mass "high" in the cusps of $\mathcal{X}$ are seen to be related as in the main theorem. More precisely, if $\mathcal{X}_{>s}$ denotes the part of $\mathcal{X}$ above height s (the notion of height is defined in Section 3 below), then

$$h_\mu(T) \leq h_\mu(T,\eta) + c_s + \tfrac{1}{2} h_m(T)\mu(\mathcal{X}_{>s})$$

with a global constant c_s such that $c_s \to 0$ as $s \to \infty$. We remark that η is independent of μ. To achieve this we use a partition of $\mathcal{X}$ into a fixed compact part, the part $\mathcal{X}_{>s}$ above height s, and the strip between the compact part and $\mathcal{X}_{>s}$. The compact part is refined into very small sets, depending on the width of the strip, such that this part and the strip do not contribute to entropy.

The entropy of μ is estimated from above using the Brin-Katok Lemma, which reduces this task to counting Bowen balls needed to cover some set of fixed positive measure. In Lemma 7.3 below we prove a non-trivial bound for this number. In order to be able to establish this result, we translate the situation to Siegel sets in G (which is possible thanks to a result of Garland and Raghunathan [GR70] on fundamental domains), and make a detailed study how nearby trajectories behave high up in the cusp.

These investigations do not use the classification of $\mathbb{R}$-rank 1 simple Lie groups. Rather we take advantage of the uniform and easy to manipulate construction of rank 1 symmetric spaces of noncompact type provided by [CDKR91] and [CDKR98] and the coordinate system of the associated Lie groups adapted to their geometry.

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2. Fundamental domains in the cusps

Let A be the maximal $\mathbb{R}$-split torus in G containing the diagonalizable element $\tilde{a}$ defining an action T on $\mathcal{X}$ via $x \mapsto x\tilde{a}$. Let $C = C_A(G)$ denote the centralizer of A in G and $\mathfrak{c}$ its Lie algebra. Let $\mathfrak{g}$ denote the Lie algebra of G. Since G is of $\mathbb{R}$-rank 1, there exists a group homomorphism $\alpha\colon A \to (\mathbb{R}_{>0}, \cdot)$ such that with

$$\mathfrak{g}_j := \left\{ X \in \mathfrak{g} \;\middle|\; \forall\, a \in A\colon\ \mathrm{Ad}_a X = \alpha(a)^{\frac{j}{2}} X \right\}, \quad j \in \{\pm 1, \pm 2\},$$

we have the direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2. \tag{1}$$

We choose the homomorphism α such that $\alpha(\tilde{a}) > 1$. The Lie algebra $\mathfrak{g}$ is the direct product of a simple Lie algebra and a compact one. Unless this simple Lie algebra is isomorphic to $\mathfrak{so}(1,n)$, the homomorphism α is then unique and (1) is the restricted root space decomposition of $\mathfrak{g}$. If the simple factor of $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(1,n)$ for some $n \in \mathbb{N}$, $n \geq 2$, then there are two choices for α. Depending on the choice, either $\mathfrak{g}_2$ or $\mathfrak{g}_1$ is trivial. In this case (1) simplifies to

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{c} \oplus \mathfrak{g}_1 \quad \text{resp.} \quad \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{c} \oplus \mathfrak{g}_2,$$

each of which is the restricted root space decomposition of $\mathfrak{g}$. The first one corresponds to the Cayley-Klein models of real hyperbolic spaces, the second one to the Poincaré models. Define $\mathfrak{n} := \mathfrak{g}_2 \oplus \mathfrak{g}_1$ and let N be the connected, simply

connected Lie subgroup of G with Lie algebra $\mathfrak{n}$. By Iwasawa decomposition, there exists a maximal compact subgroup K of G such that

$$N \times A \times K \to G, \quad (n, a, k) \mapsto nak$$

is a diffeomorphism. Let

$$M := K \cap C.$$

For any $s > 0$ we set

$$A_s := \{a \in A \mid \alpha(a) > s\}.$$

Moreover, for any $s > 0$ and any compact subset η of N we define the Siegel set

$$\Omega(s, \eta) := \eta A_s K.$$

Garland and Raghunathan provide the following result on fundamental domains for the non-cocompact lattice Γ in G.

Proposition 2.1 (Theorem 0.6 and 0.7 in [GR70])**.** *There exists $s_0 > 0$, a compact subset η_0 of N and a finite subset Ξ of G such that*

(i) $G = \Gamma\Xi\Omega(s_0, \eta_0)$,
(ii) *for all $\xi \in \Xi$, the group $\Gamma \cap \xi N \xi^{-1}$ is a cocompact lattice in $\xi N \xi^{-1}$,*
(iii) *for all compact subsets η of N the set*

$$\{\gamma \in \Gamma \mid \gamma\Xi\Omega(s_0, \eta) \cap \Omega(s_0, \eta) \neq \emptyset\}$$

is finite,
(iv) *for each compact subset η of N containing η_0, there exists $s_1 > s_0$ such that for all $\xi_1, \xi_2 \in \Xi$ and all $\gamma \in \Gamma$ with $\gamma\xi_1\Omega(s_0, \eta) \cap \xi_2\Omega(s_1, \eta) \neq \emptyset$ we have $\xi_1 = \xi_2$ and $\gamma \in \xi_1 N M \xi_1^{-1}$.*

For the rest of this article we fix $s_1 > s_0 > 0$, a compact subset η_0 of N and a finite subset Ξ of G which satisfy (i)-(iv) of Proposition 2.1 with $\eta := \eta_0$.

The elements of Ξ are a minimal set of representatives for the cusps of

$$\mathcal{X} := \Gamma\backslash G,$$

and for each $\xi \in \Xi$, the Siegel set $\xi\Omega(s_1, \eta)$ is modulo $\Gamma \cap \xi N M \xi^{-1}$ a fundamental domain for a neighborhood of the corresponding cusp of $\mathcal{X}$.

3. The height function

For each $\xi \in \Xi$, we introduce a height function which measures how far a point $x \in \mathcal{X}$ is "in the cusp ξ". More precisely, the ξ-height of x is the maximal value $\alpha(a)$ for an x-representative ξnak in $G = \xi NAK$. The maximum over all ξ-heights gives the total height of $x \in \mathcal{X}$. For a coordinate-free definition of the height functions, we introduce a representation derived from the adjoint representation. This representation was already used in [Dan84].

For each $\xi \in \Xi$ we let

$$L_\xi := \xi N M \xi^{-1}$$

and denote its Lie algebra by $\mathfrak{l}_\xi$. Set $\ell := \dim \mathfrak{l}_\xi$ (which in fact is independent of ξ) and let V be the ℓ-th exterior power of $\mathfrak{g}$

$$V := \bigwedge^\ell \mathfrak{g}.$$

Let ϱ be the right G-action on V, given by the ℓ-th exterior power of

$$\mathrm{Ad}\circ(\cdot)^{-1}\colon G\to\mathrm{End}(\mathfrak{g}),\quad g\mapsto \mathrm{Ad}_{g^{-1}},$$

hence

$$\varrho := \bigwedge^{\ell}\big(\mathrm{Ad}\circ(\cdot)^{-1}\big)\colon G\to\mathrm{End}(V).$$

We fix a non-zero element v_ξ in the one-dimensional space

$$W_\xi := \bigwedge^{\ell}\mathfrak{l}_\xi$$

and let

$$\theta_\xi\colon \xi NMA\xi^{-1}\to\mathbb{R}_{>0}$$

be the unique group homomorphism into the multiplicative group $(\mathbb{R}_{>0},\cdot)$ such that for all $g\in\xi NMA\xi^{-1}$ we have

$$v_\xi\varrho(g)=\theta_\xi(g)v_\xi.$$

One easily shows that $\theta_\xi(g)=1$ for g in the connected component of L_ξ, and

$$\theta_\xi(\xi a\xi^{-1})=\alpha(a)^{-\left(\frac{1}{2}\dim\mathfrak{g}_1+\dim\mathfrak{g}_2\right)}$$

for $a\in A$. Let

$$q:=\tfrac{1}{2}\dim\mathfrak{g}_1+\dim\mathfrak{g}_2.$$

We choose a $\varrho(K)$-invariant inner product $\langle\cdot,\cdot\rangle$ on V and denote its associated norm by $\|\cdot\|$.

For $\xi\in\Xi$, the *ξ-height* of $x\in\mathcal{X}$ is defined as

$$\mathrm{ht}_\xi(x):=\sup\left\{\left(\frac{\|v_\xi\varrho(g)\|}{\|v_\xi\varrho(\xi)\|}\right)^{-\frac{1}{q}}\ \middle|\ g\in G,\ x=\Gamma g\right\}. \tag{2}$$

If $g\in G$ is represented as $g=\xi nak$ with $n\in N$, $a\in A$ and $k\in K$, then by definition

$$\left(\frac{\|v_\xi\varrho(g)\|}{\|v_\xi\varrho(\xi)\|}\right)^{-\frac{1}{q}}=\alpha(a).$$

Hence this value only depends on the A-components of g when represented in $\xi NAK(=G)$, of which we may think as an Iwasawa decomposition of G relative to ξ.

The *height* of $x\in\mathcal{X}$ is

$$\mathrm{ht}(x):=\max\big\{\,\mathrm{ht}_\xi(x)\ \big|\ \xi\in\Xi\big\}.$$

For $s>0$ and $\xi\in\Xi$ we set

$$\mathcal{X}(\xi,s):=\{x\in\mathcal{X}\mid \mathrm{ht}_\xi(x)>s\}$$

and

$$\mathcal{X}_{>s}:=\{x\in\mathcal{X}\mid \mathrm{ht}(x)>s\}=\bigcup_{\xi\in\Xi}\mathcal{X}(\xi,s). \tag{3}$$

In the following we will see that the points in $\mathcal{X}(\xi,s)$ correspond to the elements in the Siegel set $\xi\Omega(s,\eta)$. To that end let B_δ denote the open $\|\cdot\|$-ball in V with radius $\delta>0$, centered at 0. We define

$$\delta_\xi(s):=s^{-q}\|v_\xi\varrho(\xi)\|.$$

Proposition 3.1 (Corollary 2.3 in [Dan84])**.** *Let $\xi \in \Xi$, $s > 0$, and $g \in G$. Then $\Gamma g \in \Gamma\backslash\Gamma\xi\Omega(s,\eta)$ if and only if $v_\xi \varrho(\gamma g) \in B_{\delta_\xi(s)}$ for some $\gamma \in \Gamma$. Further, if $s \geq s_1$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $v_\xi \varrho(\gamma_j g) \in B_{\delta_\xi(s)}$ for $j = 1, 2$, then $v_\xi \varrho(\gamma_1 g) \in \{\pm v_\xi \varrho(\gamma_2 g)\}$.*

Thus

$$\mathcal{X}(\xi, s) = \Gamma\backslash\Gamma\xi\Omega(s,\eta)$$

for all $\xi \in \Xi$ and $s > 0$. If $s \geq s_1$, the supremum in (2) is attained. Moreover, by Proposition 2.1(iv),

$$\mathcal{X}(\xi, s) \cap \mathcal{X}(\xi', s) = \emptyset$$

if $\xi \neq \xi' \in \Xi$. Hence the sets $\mathcal{X}(\xi, s)$ are then disjoint neighborhoods of the cusps of $\mathcal{X}$, and the union in (3) is disjoint.

4. Coordinate system for G

The Lie algebra $\mathfrak{g}$ is the direct sum of a simple Lie algebra of rank 1 and a compact one. Since the height function is right-$\varrho(K)$-invariant and all further considerations are right-$\varrho(K)$-invariant, we can restrict to $\mathfrak{g}$ being simple. [CDKR91] and [CDKR98] provide a classification-free construction of all Riemannian symmetric spaces of noncompact type and rank one. Basal to this is the choice of a certain coordinate system for real simple Lie groups G of real rank 1, which allows to treat all these groups without refering to their classification. In the following we recall this coordinate system, the one for the associated symmetric spaces and some essential formulas.

The semidirect product NA is parametrized by

$$\mathbb{R}_{>0} \times \mathfrak{g}_2 \times \mathfrak{g}_1 \to NA, \quad (s, Z, X) \mapsto \exp(Z + X) \cdot a_s$$

such that $\alpha(a_s) = s$. The action of $a_s = (s, 0, 0) \in A$ on $n = (1, Z, X) \in N$ is then given by

$$a_s n = (s, sZ, s^{1/2}X).$$

We define an inner product on $\mathfrak{n} = \mathfrak{g}_2 \oplus \mathfrak{g}_1$ as follows. Let $\mathfrak{k}$ be the Lie algebra of K. Let θ be a Cartan involution of $\mathfrak{g}$ such that $\mathfrak{k}$ is its 1-eigenspace. For $X, Y \in \mathfrak{n}$ we define

$$\langle X, Y\rangle := -\frac{1}{\dim \mathfrak{g}_1 + 4\dim \mathfrak{g}_2} B(X, \theta Y)$$

where B is the Killing form of $\mathfrak{g}$. It is well-known that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathfrak{n}$. As in [CDKR91, CDKR98], we identify $G/K \cong NA \cong \mathbb{R}_{>0} \times \mathfrak{g}_2 \times \mathfrak{g}_1$ with

$$D := \left\{ (t, Z, X) \in \mathbb{R} \times \mathfrak{g}_2 \times \mathfrak{g}_1 \;\middle|\; t > \frac{1}{4}|X|^2 \right\}$$

via

$$\begin{cases} \mathbb{R}_{>0} \times \mathfrak{g}_2 \times \mathfrak{g}_1 & \to \quad D \\ \quad (t, Z, X) & \mapsto \quad (t + \frac{1}{4}|X|^2, Z, X). \end{cases}$$

The action of an element $s = (t_s, Z_s, X_s) \in NA$ on a point $p = (t_p, Z_p, X_p) \in D$ becomes

$$s.p = \big(t_s t_p + \tfrac{1}{4}|X_s|^2 + \tfrac{1}{2} t_s^{1/2}\langle X_s, X_p\rangle, Z_s + t_s Z_p + \tfrac{1}{2} t_s^{1/2}[X_s, X_p], X_s + t_s^{1/2} X_p\big).$$

These coordinates of G/K enable us to use [CDKR91, CDKR98], and they simplify some of the expressions below, in particular the one for the geodesic inversion. To state the geodesic inversion, we define the linear map $J\colon \mathfrak{g}_2 \to \mathrm{End}(\mathfrak{g}_1)$, $Z \mapsto J_Z$, via

$$\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle \quad \text{for all } X, Y \in \mathfrak{g}_1.$$

Then the geodesic inversion σ of D at $o := (1,0,0)$ is given by (see [CDKR98])

$$\sigma(t,Z,X) = \frac{1}{t^2+|Z|^2}\big(t, -Z, (-t+J_Z)X\big).$$

We identify σ with the element in K which acts as geodesic inversion. Then G has the Bruhat decomposition ([CDKR98, Theorem 6.4])

$$G = NAM \cup NAM\sigma N.$$

Multiplying this with $\xi \in \Xi$ from the left and σ from the right, we get

$$G = \xi NAM\sigma \cup \xi NAMU$$

with $U := \sigma N \sigma$. This decomposition provides a coordinate system on G adapted to the cusp ξ. The set $\xi NAM\sigma$ we call the *small ξ-Bruhat cell* and $\xi NAMU$ the *big ξ-Bruhat cell*.

The group M is parametrized by the pairs (φ,ψ) consisting of orthogonal endomorphisms φ on $\mathfrak{g}_2$ resp. ψ on $\mathfrak{g}_1$ such that $\psi(J_Z X) = J_{\varphi(Z)}\psi(X)$ for all $(Z,X) \in \mathfrak{g}_2 \times \mathfrak{g}_1$. The action of $(\varphi,\psi) \in M$ on $p = (t,Z,X) \in D$ is given by

$$(\varphi,\psi).p = (t, \varphi(Z), \psi(X)).$$

By [CDKR98, Proposition 7.1], $|J_Z X| = |Z||X|$ for all $Z \in \mathfrak{g}_2$, $X \in \mathfrak{g}_1$.

5. Variation of height

Suppose the point $x \in \mathcal{X}$ is of big height and its trajectory stays far out for some time. In this section, we provide non-trivial bounds on the unstable components of a group element $g \in G$ representing x. In Proposition 6.3 below, this bound culminates into contraints on the perturbation allowed of x without destroying the qualitative behavior of its trajectory during this time.

Lemma 5.1. *Let $a_t, a_r \in A$, $m \in M$ and $n \in N$ with $n = (1,Z,X)$ such that $\sigma m n a_t \in N a_r K$. Then*

$$r = \frac{t}{\big(t+\frac{1}{4}|X|^2\big)^2 + |Z|^2}.$$

Proof. By Iwasawa decomposition we know that $\sigma m n a_t = n' a_r k$ for suitable $n' \in N$, $k \in K$ and $r \in \mathbb{R}_{>0}$. Suppose that $m = (\varphi,\psi)$. Applying both $\sigma m n a_t$ and $n' a_r k$ to the base point $o = (1,0,0)$ in D, we find

$$\sigma m n a_t \cdot o = n' a_r k \cdot o = n' a_r \cdot o.$$

In the coordinates of D one easily calculates that

$$\sigma m n a_t \cdot o = \\ = \frac{1}{\big(t+\frac{1}{4}|X|^2\big)^2 + |Z|^2}\left(t+\tfrac{1}{4}|X|^2, -\varphi(Z), \left(-t-\tfrac{1}{4}|X|^2 + J_{\varphi(Z)}\right)\psi(X)\right).$$

Suppose that $n' = (1, Z', X')$. Then

$$n' a_r \cdot o = \left(r + \tfrac{1}{4}|X'|^2, Z', X'\right).$$

Thus

$$\begin{aligned} X' &= \frac{1}{\left(t + \frac{1}{4}|X|^2\right)^2 + |Z|^2} \left(-t - \tfrac{1}{4}|X|^2 + J_{\varphi(Z)}\right) \psi(X), \\ |X'|^2 &= \frac{1}{\left(\left(t + \frac{1}{4}|X|^2\right)^2 + |Z|^2\right)^2} \Big(\left(t + \tfrac{1}{4}|X|^2\right)^2 |X|^2 + \left|J_{\varphi(Z)}\psi(X)\right|^2 \\ &\qquad\qquad - 2\left(t + \tfrac{1}{4}|X|^2\right) \left\langle \psi(X), J_{\varphi(Z)}\psi(X)\right\rangle \Big) \\ &= \frac{|X|^2}{\left(t + \frac{1}{4}|X|^2\right)^2 + |Z|^2}, \end{aligned}$$

and

$$r = \frac{t + \frac{1}{4}|X|^2}{\left(t + \frac{1}{4}|X|^2\right)^2 + |Z|^2} - \tfrac{1}{4}|X'|^2 = \frac{t}{\left(t + \frac{1}{4}|X|^2\right)^2 + |Z|^2}. \qquad \square$$

Lemma 5.2. *Let $\xi \in \Xi$ and $g \in G$. If $g = \xi n a_s m \sigma$ with $n \in N$ and $m \in M$, then*

$$\left(\frac{\|v_\xi \varrho(g a_t)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}} = \frac{s}{t}.$$

If $g = \xi n a_s m \sigma (1, Z, X) \sigma$ with $n \in N$ and $m \in M$, then

$$\left(\frac{\|v_\xi \varrho(g a_t)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}} = s \cdot \frac{\frac{1}{t}}{\left(\frac{1}{t} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}.$$

Let us note that the first case corresponds to a trajectory pointing straight out of the cusp ξ. In the second case, the element $u = \sigma(1, Z, X)\sigma$ determines the perturbation to the trajectory pointing straight into the cusp ξ. If $(Z, X) = (0, 0)$, the second case correspond to a trajectory pointing straight into the cusp ξ, and the formula simplifies to

$$\left(\frac{\|v_\xi \varrho(g a_t)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}} = st.$$

The effect of the perturbation for $(Z, X) \neq (0, 0)$ is given by the factor

$$\frac{t^{-2}}{\left(\frac{1}{t} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}.$$

Proof of Lemma 5.2. At first we suppose that $g = \xi n a_s m \sigma$. Then

$$g a_t = \xi n a_{s/t} m \sigma = \xi n a_{s/t} \xi^{-1} \xi m \sigma \quad \in \left(\xi N A \xi^{-1}\right)\left(\xi K\right).$$

Hence

$$\|v_\xi \varrho(g a_t)\| = \theta_\xi(\xi a_{s/t} \xi^{-1}) \|v_\xi \varrho(\xi)\| = \left(\frac{s}{t}\right)^{-q} \|v_\xi \varrho(\xi)\|.$$

Suppose now that $g = \xi n m a_s u$ with $u = \sigma n' \sigma$ and $n' = (1, Z, X)$. Then (for some $m' \in M$)

$$\begin{aligned}\|v_\xi \varrho(g a_t)\| &= s^{-q}\|v_\xi \varrho(\xi \sigma m' n' \sigma a_t)\| = s^{-q}\|v_\xi \varrho(\xi \sigma m' n' a_{1/t}\sigma)\| \\ &= s^{-q}\|v_\xi \varrho(\xi \sigma m' n' a_{1/t})\|.\end{aligned}$$

Lemma 5.1 yields

$$\sigma m' n' a_{1/t} = n'' a_r k$$

for some $n'' \in N$, $k \in K$ and

$$r = \frac{\frac{1}{t}}{\left(\frac{1}{t} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}.$$

Thus,

$$\|v_\xi \varrho(g a_t)\| = \left(\frac{\frac{1}{t}}{\left(\frac{1}{t} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}\right)^{-q} s^{-q}\|v_\xi \varrho(\xi)\|.$$

□

The following proposition describes the amount of time a trajectory spends in a neighborhood of the cusp ξ.

Proposition 5.3. *Let $\xi \in \Xi$ and $g \in G$. Write $\delta := \|v_\xi \varrho(g)\|$. If $g \in \xi N A M \sigma$, then $v_\xi \varrho(g a_t) \in B_\delta$ if and only if $t < 1$. If $g = \xi n a_s m u \in \xi N A M U$ with $u = \sigma(1, Z, X)\sigma$, then $v_\xi \varrho(g a_t) \in B_\delta$ if and only if*

$$t \in \left(\frac{1}{\frac{1}{16}|X|^4 + |Z|^2}, 1\right) \cup \left(1, \frac{1}{\frac{1}{16}|X|^4 + |Z|^2}\right).$$

If $u = \mathrm{id}$, then $\left(\frac{1}{16}|X|^4 + |Z|^2\right)^{-1}$ is to be understood as ∞.

Proof. The first part of the statement follows immediately from Lemma 5.2. Suppose now that $g = \xi n m a_s u$ with $u = \sigma n' \sigma$ and $n' = (1, Z, X)$. By Lemma 5.2,

$$\|v_\xi \varrho(g a_r)\| = \left(\frac{\frac{1}{r}}{\left(\frac{1}{r} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}\right)^{-q} s^{-q}\|v_\xi \varrho(\xi)\|. \tag{4}$$

Applying (4) for $r = 1$ and $r = t$, we see that

$$\|v_\xi \varrho(g a_t)\| < \|v_\xi \varrho(g)\|$$

if and only if

$$\frac{1}{\left(1 + \frac{1}{4}|X|^2\right)^2 + |Z|^2} < \frac{\frac{1}{t}}{\left(\frac{1}{t} + \frac{1}{4}|X|^2\right)^2 + |Z|^2},$$

which is equivalent to

$$\left(1 - \frac{1}{t}\right)\left(-\frac{1}{t} + \frac{1}{16}|X|^4 + |Z|^2\right) < 0.$$

This is the case if and only if

$$|Z|^2 + \frac{1}{16}|X|^4 < \frac{1}{t} < 1 \quad \text{or} \quad |Z|^2 + \frac{1}{16}|X|^4 > \frac{1}{t} > 1.$$

□

Suppose that $\|v_\xi\varrho(ga_t)\| = \|v_\xi\varrho(\gamma g a_t)\|$ for some $g \in G$, $\gamma \in \Gamma$ and all t in a non-trivial interval (ie., an interval which contains at least two points). Then Lemma 5.2 yields that g and γg have the same A-component in ξNAK and they are in the same ξ-Bruhat cell. If moreover, g and γg are in the big ξ-Bruhat cell, then also the norms of their U-components are equal. The following lemma shows that far out in the cusp much more is true.

Lemma 5.4. *Let $\xi \in \Xi$ and suppose that $g \in G$ and $\gamma \in \Gamma$ are such that*

$$\|v_\xi\varrho(g)\| = \|v_\xi\varrho(\gamma g)\| < \delta_\xi(s_1).$$

Then $\gamma \in \xi NM\xi^{-1}$. In particular, if $g = \xi nam\sigma$ resp. $g = \xi namu$ with $n \in N$, $a \in A$, $m \in M$ and $u \in U$, then $\gamma g = \xi n'am'\sigma$ resp. $\gamma g = \xi n'am'u$ for some $n' \in N$, $m' \in M$.

Proof. By [Dan84, Lemma 2.2] (see also Proposition 3.1), for each $s > 0$ we have

$$L_\xi \xi A_s K = \left\{ g \in G \,\middle|\, v_\xi\varrho(g) \in B_{\delta_\xi(s)} \right\}.$$

Hence $g, \gamma g \in L_\xi\xi A_{s_1}K$. By [Dan84, Remark 1.3] (with η as in Proposition 2.1),

$$L_\xi\xi A_{s_1}K = (\Gamma \cap L_\xi)\xi\eta A_{s_1}K.$$

Hence there exist $\gamma_1, \gamma_2 \in \Gamma \cap L_\xi$, $h_1, h_2 \in \eta A_{s_1}K$ such that

$$g = \gamma_1\xi h_1, \quad \gamma g = \gamma_2 \xi h_2.$$

Therefore

$$g \in \gamma_1\xi\Omega(s_1,\eta) \cap \gamma^{-1}\gamma_2\xi\Omega(s_1,\eta).$$

Proposition 2.1(iv) yields $\gamma_1^{-1}\gamma^{-1}\gamma_2 \in \xi NM\xi^{-1}$. Thus, $\gamma \in \xi NM\xi^{-1}$. □

For the proof of the following proposition we recall that the supremum in the definition of ξ-height (2) is realized if $\mathrm{ht}_\xi(x) \geq s_1$.

Proposition 5.5. *Let $s > s_1$ and $x \in \mathcal{X}$. Suppose that there exists an interval I in $\mathbb{R}$ such that $\mathrm{ht}(xa_t) > s$ for all $t \in I$. Then there exists a unique cusp $\xi \in \Xi$ and a (non-unique) element $g \in G$ with $x = \Gamma g$ such that*

$$\mathrm{ht}(xa_t) = \mathrm{ht}_\xi(xa_t) = \left(\frac{\|v_\xi\varrho(ga_t)\|}{\|v_\xi\varrho(\xi)\|}\right)^{-\frac{1}{q}}$$

for all $t \in I$. Moreover, if $1 \in I$ and if there exists $t \in I$ with $t > 1$ and $\mathrm{ht}(xa_t) > \mathrm{ht}(x)$, then $g = \xi n a_r m u$ for some $r > 0$, $n \in N$, $m \in M$ and $u \in U$. The elements a_r and u do not depend on the choice of g. Finally, if $u = \sigma(1, Z, X)\sigma$, then

$$|X| < 2t^{-1/4} \quad \textit{and} \quad |Z| < t^{-1/2}.$$

Proof. If $y \in \mathcal{X}$ and $\xi \in \Xi$ such that $\mathrm{ht}_\xi(y) > s_1$, then there exists $h \in G$ such that $y = \Gamma h$ and

$$\mathrm{ht}_\xi(y) = \left(\frac{\|v_\xi\varrho(h)\|}{\|v_\xi\varrho(\xi)\|}\right)^{-\frac{1}{q}}.$$

Since the function

$$\begin{cases} \mathbb{R}_{>0} & \to & \mathbb{R} \\ r & \mapsto & \|v_\xi\varrho(ga_r)\| \end{cases}$$

is continuous, there exists an open neighborhood J of 1 in $\mathbb{R}_{>0}$ such that $\mathrm{ht}_\xi(ya_r) > s_1$ for all $r \in J$. For $\xi \in \Xi$ let

$$J_\xi := \{t \in I \mid \mathrm{ht}_\xi(xa_t) > s\}.$$

These sets are pairwise disjoint, open in I and cover I. Since I is connected, there exists a unique $\xi \in \Xi$ with $I = J_\xi$. Thus

$$\mathrm{ht}(xa_t) = \mathrm{ht}_\xi(xa_t)$$

for all $t \in I$. For each $t \in I$ pick an element $g_t \in G$ such that $x = \Gamma g_t$ and

$$\mathrm{ht}_\xi(xa_t) = \left(\frac{\|v_\xi \varrho(g_t a_t)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}}.$$

Let J_t be the set of $p \in I$ such that

$$\mathrm{ht}_\xi(xa_p) = \left(\frac{\|v_\xi \varrho(g_t a_p)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}}.$$

Then I is covered by the sets J_t, and these are open in I by Proposition 3.1. If J_t and J_r overlap for some $t, r \in I$, $t \neq r$, then Lemma 5.4 and 5.2 imply that $J_t = J_r$. In turn, $J_t = I$ for each $t \in I$.

The remaining statements follow immediately from Proposition 5.3 and Lemma 5.4. $\square$

6. Common cusp excursions of nearby points

For $s > 0$ we define

$$\mathcal{X}_{\leq s} := \mathcal{X} \setminus \mathcal{X}_{>s}.$$

Further we let

$$r_0 := \alpha(\tilde{a}) \quad > 1.$$

Each connected component of $\mathcal{X}$ of height above s_1 can essentially be identified with a Siegel set (cf. Proposition 2.1). For the proof of the main theorem, trajectories of points $x \in \mathcal{X}$ are only considered time-discretized by the map T. In the following lemma we construct a height level s_2 above which we can identify pieces of these discretized trajectories with trajectory segments in a Siegel set. More specifically, as soon as we know that two consecutive points of the discretized trajectory stay above height $s \geq s_2$, then the (continuous) trajectory segment of the corresponding geodesic also stays above height s and, in particular, does not visit the compact set $\mathcal{X}_{\leq s_1}$. Then we construct a second height level $s_3 > s_2$ such that any discretized trajectory entering $\mathcal{X}_{>s_3}$ can locally be identified with a continuous trajectory segment in the Siegel set. In Section 8 below this will be crucial to effectively determine the behavior of nearby starting trajectories. Of special importance for Section 7 below is the point (v) of the following lemma, which states that if we start to descent somewhere high in a cusp, then we actually descent up to below height s_3.

Lemma 6.1. *There exist $s_3 > s_2 > s_1$ such that we have the following properties:*

(i) *If $x \in \mathcal{X}_{>s_2}$, then $\mathrm{ht}(xa_t) > 2s_1$ for all $t \in [r_0^{-1}, r_0]$.*
(ii) *If $s \geq s_2$ and $x, Tx \in \mathcal{X}_{>s}$, then $\mathrm{ht}(xa_t) > s$ for all $t \in [1, r_0]$.*

(iii) *Let $s > s_3$. If $x \in \mathcal{X}_{\le s_3}$ and $T^j x \in \mathcal{X}_{>s}$ for some $j \in \mathbb{N}$, then there exists $n \in \{0, \dots, j-1\}$ such that $\mathrm{ht}(T^n x) \le s_3$ and $\mathrm{ht}(xa_t) > s_2$ for all $t \in [r_0^n, r_0^j]$.*

(iv) *Let $s > s_3$. If $x \in \mathcal{X}_{>s}$ and $T^j x \in \mathcal{X}_{\le s_3}$ for some $j \in \mathbb{N}$, then there exists $n \in \{1, \dots, j\}$ such that $\mathrm{ht}(T^n x) \le s_3$ and $\mathrm{ht}(xa_t) > s_2$ for all $t \in [1, r_0^n]$.*

(v) *Let $s > s_3$. If $x \in \mathcal{X}_{>s}$ and $Tx \in \mathcal{X}_{\le s}$, then there exists $n \in \mathbb{N}$ such that $T^n x \in \mathcal{X}_{\le s_3}$ and $T^k x \in \mathcal{X}_{\le s}$ for all $k = 1, \dots, n$.*

Proof. Choose $s_4 > 2r_0^2 s_1$. Let $x \in \mathcal{X}_{>s_4}$. At first we prove that $xa_t \in \mathcal{X}_{>2s_1}$ for all $t \in [r_0^{-1}, r_0]$. Since $s_4 > s_1$, there exist a unique $\xi \in \Xi$ and an element $g \in G$ such that $x = \Gamma g$ and

$$\mathrm{ht}(x) = \mathrm{ht}_\xi(x) = \left(\frac{\|v_\xi \varrho(g)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}}.$$

For all $t \in [r_0^{-1}, r_0]$ we have

$$\mathrm{ht}(xa_t) \ge \mathrm{ht}_\xi(xa_t) \ge \left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}}.$$

In the following we show that

$$\left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}} > 2s_1$$

for all $t \in [r_0^{-1}, r_0]$. To that end we distinguish two cases for the form of g.

At first suppose that $g = \xi n a_s m \sigma$ with $n \in N$, $m \in M$. Then (see Lemma 5.2)

$$\left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}} = \frac{s}{t}.$$

From

$$\left(\frac{\|v_\xi \varrho(g)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}} = s > s_4$$

it follows that

$$\left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}} > \frac{s_4}{t} \ge \frac{s_4}{r_0} > 2s_1$$

for all $t \in [r_0^{-1}, r_0]$. Now suppose that $g = \xi n a_s m \sigma(1, Z, X)\sigma$ with $n \in N$, $m \in M$. Lemma 5.2 states that

$$\left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}} = s \cdot \frac{\frac{1}{t}}{\left(\frac{1}{t} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}.$$

Since

$$\left(\frac{\|v_\xi \varrho(g)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}} = s \cdot \frac{1}{\left(1 + \frac{1}{4}|X|^2\right)^2 + |Z|^2} > s_4,$$

we have

$$\left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}} > s \cdot \frac{\frac{1}{t}}{\left(1 + \frac{1}{4}|X|^2\right)^2 + |Z|^2} > \frac{s_4}{t} > 2s_1$$

for all $t \in [1, r_0]$. Further one easily shows that

$$\frac{\left(1+\frac{1}{4}|X|^2\right)^2+|Z|^2}{\left(r_0+\frac{1}{4}|X|^2\right)^2+|Z|^2} > \frac{1}{r_0^2}.$$

For all $t \in [r_0^{-1}, 1]$ it follows that

$$\begin{aligned}\left(\frac{\|v_\xi\varrho(ga_t)\|}{\|v_\xi\varrho(\xi)\|}\right)^{-\frac{1}{q}} &= \frac{1}{t}\cdot\frac{s}{\left(\frac{1}{t}+\frac{1}{4}|X|^2\right)^2+|Z|^2} > \frac{s_4}{t}\cdot\frac{\left(1+\frac{1}{4}|X|^2\right)^2+|Z|^2}{\left(\frac{1}{t}+\frac{1}{4}|X|^2\right)^2+|Z|^2}\\ &\geq s_4\cdot\frac{\left(1+\frac{1}{4}|X|^2\right)^2+|Z|^2}{\left(r_0+\frac{1}{4}|X|^2\right)^2+|Z|^2} \geq \frac{s_4}{r_0^2} > 2s_1.\end{aligned}$$

This proves the claim. We pick any $s_2 > s_4$. Then property (i) follows.

We now show that (ii) is satisfied als well. So suppose that $s \geq s_2$ and $x, Tx \in \mathcal{X}_{>s}$. From (i) it follows that

$$\mathrm{ht}(xa_t) > 2s_1$$

for all $t \in [1, r_0]$. Thus, by Proposition 5.5 we find a unique $\xi \in \Xi$ and some $g \in G$ such that $\Gamma g = x$ and

$$\mathrm{ht}(xa_t) = \left(\frac{\|v_\xi\varrho(ga_t)\|}{\|v_\xi\varrho(\xi)\|}\right)^{-\frac{1}{q}}$$

for all $t \in [1, r_0]$. We will show that $\mathrm{ht}(xa_t) \geq \mathrm{ht}(x)$ or $\mathrm{ht}(xa_t) \geq \mathrm{ht}(Tx)$ for all $t \in [1, r_0]$, which implies (ii). Suppose first that $g = \xi n a_r m \sigma$ for some $n \in N$, $r > 0$ and $m \in M$. By Lemma 5.2 we have

$$\mathrm{ht}(xa_t) = \frac{r}{t}$$

for all $t \in [1, r_0]$. Thus,

$$\mathrm{ht}(xa_t) \geq \mathrm{ht}(Tx) \quad \text{for all } t \in [1, r_0].$$

Suppose now that $g = \xi n a_r m \sigma (1, Z, X)\sigma$ for some $n, (1, Z, X) \in N$, $r > 0$ and $m \in M$. By Lemma 5.2

$$\mathrm{ht}(xa_t) = r\cdot\frac{t^{-1}}{\left(t^{-1}+\frac{1}{4}|X|^2\right)^2+|Z|^2}.$$

Let $t \in (1, r_0]$. Straightforward calculations show that $\mathrm{ht}(xa_t) \geq \mathrm{ht}(x)$ if and only if

$$t \leq \frac{1}{\frac{1}{16}|X|^4+|Z|^2}. \tag{5}$$

Thus, in particular, if we suppose $\mathrm{ht}(Tx) \geq \mathrm{ht}(x)$, then

$$r_0 \leq \frac{1}{\frac{1}{16}|X|^4+|Z|^2},$$

and, *a fortiori*, (5) is satisfied for all $t \in (1, r_0]$. Analogously, for $t \in [1, r_0)$, $\mathrm{ht}(xa_t) \geq \mathrm{ht}(Tx)$ if and only if

$$t \geq \frac{r_0^{-1}}{\frac{1}{16}|X|^4+|Z|^2}. \tag{6}$$

Assuming $\mathrm{ht}(Tx) \geq \mathrm{ht}(x)$ implies

$$1 \geq \frac{r_0^{-1}}{\frac{1}{16}|X|^4 + |Z|^2},$$

and thus (6) is fullfilled for all $t \in [1, r_0)$. This proves (ii).

Choose $s_3 > r_0^2 s_2$. Let $s > s_3$ and suppose that $x \in \mathcal{X}_{>s}$ and $Tx \in \mathcal{X}_{\leq s}$. Our previous consideration (with s_2 instead of $2s_1$, and s_3 instead of s_2) shows that $Tx \in \mathcal{X}_{>s_2}$. Let $n \in \mathbb{N}$ be maximal such that $T^k x \in \mathcal{X}_{>s_2}$ for $k = 1, \ldots, n$. In the following we prove that $T^n x \in \mathcal{X}_{\leq s_3}$ and $T^k x \in \mathcal{X}_{\leq s}$ for $k = 1, \ldots, n$. By our previous consideration, $\mathrm{ht}(xa_t) > s_2 (> s_1)$ for all $t \in [1, r_0^n]$. Proposition 5.5 shows that there exist a unique $\xi \in \Xi$ and an element $g \in G$ such that $x = \Gamma g$ and

$$\mathrm{ht}(xa_t) = \mathrm{ht}_\xi(xa_t) = \left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}}$$

for all $t \in [1, r_0^n]$. If $g = \xi n a_s m \sigma$ with $n \in N$ and $m \in M$, then the map $t \mapsto \mathrm{ht}(xa_t)$ is strictly monotone decreasing on the interval $[1, r_0^n]$. Hence, in this case, $T^k x \in \mathcal{X}_{\leq s}$ for all $k = 1, \ldots, n$. If $g = \xi n a_s m \sigma(1, Z, X)\sigma$ with $n \in N$ and $m \in M$, then the map $t \mapsto \mathrm{ht}(xa_t)$ is strictly monotone increasing on

$$\left[1, \left(\tfrac{1}{16}|X|^4 + |Z|^2\right)^{-1/2}\right)$$

and strictly monotone decreasing on

$$I := \left(\left(\tfrac{1}{16}|X|^4 + |Z|^2\right)^{-1/2}, r_0^n\right].$$

From $\mathrm{ht}(Tx) < \mathrm{ht}(x)$ it follows that $r_0 \in I$. Hence $T^k x \in \mathcal{X}_{\leq s}$ for all $k = 1, \ldots, n$. Assume that $T^n x \in \mathcal{X}_{>s_3}$. The proof of (i) yields $T^{n+1} x \in \mathcal{X}_{>s_2}$, which contradicts to the definition of n. Thus $T^n x \in \mathcal{X}_{\leq s_3}$. This proves (v) and, in combination with (i) and (ii), also (iv).

Suppose that $x \in \mathcal{X}_{>s}$ and $T^{-1}x \in \mathcal{X}_{\leq s}$. Let $n \in \mathbb{N}$ be maximal such that $T^{-k}x \in \mathcal{X}_{>s_2}$ for $k = 1, \ldots, n$. As before we see that $T^{-n}x \in \mathcal{X}_{\leq s_3}$. Then (iii) follows immediately. $\square$

Given a point $x \in \mathcal{X}$ whose orbit stays near the cusp ξ for the next S steps, Proposition 6.3 below provides non-trivial contraints on small perturbations of x which do not destroy the qualitative behavior of the orbit for these next S steps. The following lemma is needed for its proof.

Lemma 6.2. *Let D^U be a bounded subset of U. Let $\xi \in \Xi$ and $g = \xi n a_r m u \in G$ with $n \in N$, $a_r \in A$, $m \in M$ and $u = \sigma(1, Z, X)\sigma \in D^U$. Suppose that*

$$\left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}} > \lambda \left(\frac{\|v_\xi \varrho(g)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}}$$

for some $t > 1$ and $\lambda > 0$. Then there exist $c_1, c_2 > 0$, only depending on D^U and λ, such that

$$|X| < c_1 t^{-1/4} \quad \text{and} \quad |Z| < c_2 t^{-1/2}.$$

Proof. For $\lambda \geq 1$, the statement is already proven in Proposition 5.5. So suppose $1 > \lambda > 0$. Invoking Lemma 5.2 we find

$$t\left[\left(\frac{1}{t}+\frac{1}{4}|X|^2\right)^2+|Z|^2\right] < \frac{1}{\lambda}\left[\left(1+\frac{1}{4}|X|^2\right)^2+|Z|^2\right]. \tag{7}$$

Thus,

$$t\left(\frac{1}{t}+\frac{1}{4}|X|^2\right)^2 < \frac{1}{\lambda}\left(1+\frac{1}{4}|X|^2\right)^2+(\lambda^{-1}-t)|Z|^2.$$

For $t > \lambda^{-1}$, it follows

$$t\left(\frac{1}{t}+\frac{1}{4}|X|^2\right)^2 < \lambda^{-1}\left(1+\frac{1}{4}|X|^2\right)^2.$$

Therefore,

$$|X|^2 < 4\frac{(t\lambda^{-1})^{\frac{1}{2}}-1}{t^{\frac{1}{2}}-\lambda^{-\frac{1}{2}}}t^{-\frac{1}{2}}.$$

Hence, for $t > \lambda^{-1}+\varepsilon$, any (fixed) $\varepsilon > 0$, we have

$$|X| < c_1 t^{-\frac{1}{4}}$$

for some constant $c_1 > 0$. Since $|X|$ is bounded, by possibly choosing a larger c_1, this estimate holds for all $t > 1$. To deduce the bound for $|Z|$ we note that (7) yields

$$\begin{aligned}(t-\lambda^{-1})|Z|^2 &< \lambda^{-1}\left(1+\frac{1}{4}|X|^2\right)^2 - t\left(\frac{1}{t}+\frac{1}{4}|X|^2\right)^2\\ &< \lambda^{-1}+\frac{c_1^2}{2}(\lambda^{-1}-1)t^{-\frac{1}{2}}+\frac{c_1^4}{16}\lambda^{-1}t^{-\frac{1}{4}}-t^{-1}-\frac{c_1^4}{16}t^{\frac{3}{4}}.\end{aligned}$$

Suppose that $t > \lambda^{-1}+\varepsilon$ for some $\varepsilon > 0$. Then

$$|Z|^2 < \frac{c_1' t^{\frac{3}{4}}+c_2' t^{-\frac{1}{2}}+c_3' t^{-\frac{1}{4}}+c_4' t^{-1}+c_5'}{1-(t\lambda)^{-1}}t^{-1}$$

with some constants $c_1', \dots, c_5'$ such that $c_1' < 0$. The factor before t^{-1} is bounded. Thus,

$$|Z| < c_2 t^{-\frac{1}{2}}$$

for some constant $c_2 > 0$. As before, since $|Z|$ is bounded, this estimate holds for all $t > 1$ after possibly choosing a larger c_2. This completes the proof. □

Let d be any left-G-invariant metric on G. For $r > 0$ let B_r^G denote the open d-ball in G centered at the identity of G. For $\kappa > 0$ let D_κ^U denote the subset of U consisting of the elements $u = \sigma(1, Z, X)\sigma$ with $|Z| < \kappa$ and $|X| < \kappa$, and let $D_\kappa^{NAM} := B_\kappa^G \cap NAM$. Further let

$$D_\kappa := D_\kappa^U D_\kappa^{NAM}. \tag{8}$$

Clearly, D_κ is open. We choose $\kappa > 0$ such that for all $h \in D_\kappa$

$$\|\varrho(h)\|, \|\varrho(h^{-1})\| \leq \left(\frac{s_1}{s_2}\right)^{-q}. \tag{9}$$

Proposition 6.3. *There exist $c_3, c_4 > 0$ such that the following holds: Let $x \in \mathcal{X}$, $S \in \mathbb{N}$, $h \in D_\kappa^U D_\kappa^{NAM}$ be such that $\mathrm{ht}(T^j x) > s_2$ and $\mathrm{ht}(T^j(xh)) > s_2$ for $j = 0, \ldots, S$, $\mathrm{ht}(T^S x) > \mathrm{ht}(x)$ and $\mathrm{ht}(T^S(xh)) > \mathrm{ht}(xh)$. Suppose that $h = \sigma(1, Z, X)\sigma n a_r m$. Then*

$$|X| \le c_3 r_0^{-S/4} \quad \textit{and} \quad |Z| \le c_4 r_0^{-S/2}.$$

Proof. By Lemma 6.1 we have $\mathrm{ht}(xa_t) > s_2$ and $\mathrm{ht}(xha_t) > s_2$ for all $t \in [1, r_0^S]$. Since $s_2 > s_1$, Proposition 5.5 shows that there exist a unique cusp $\xi \in \Xi$ and an element $g \in G$ such that $x = \Gamma g$ and

$$\mathrm{ht}(xa_t) = \left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}}$$

for all $t \in [1, r_0^S]$. Moreover, there exist a unique cusp $\xi_1 \in \Xi$ and an element $g_1 \in G$ such that $xh = \Gamma g_1 h$ and

$$\mathrm{ht}(xha_t) = \left(\frac{\|v_{\xi_1} \varrho(g_1 h a_t)\|}{\|v_{\xi_1} \varrho(\xi_1)\|} \right)^{-\frac{1}{q}}$$

for all $t \in [1, r_0^S]$. In the following we show that $\xi = \xi_1$ and that we can choose $g_1 = g$. We have

$$\|v_\xi \varrho(gha_t)\| = \|v_\xi \varrho(ga_t a_{t^{-1}} h a_t)\| \le \|v_\xi \varrho(ga_t)\| \cdot \|\varrho(a_{t^{-1}} h a_t)\|.$$

Now, $a_{t^{-1}} h a_t \in D_\kappa$ for t near 1, say in the non-trivial interval I. By (9), for $t \in I$ this yields

$$\|\varrho(a_{t^{-1}} h a_t)\| \le \left(\frac{s_1}{s_2} \right)^{-q}.$$

Thus, for $t \in I$,

$$\begin{aligned} \|v_\xi \varrho(gha_t)\| &\le \|v_\xi \varrho(ga_t)\| \left(\frac{s_1}{s_2} \right)^{-q} < \left(\frac{s_1}{s_2} \right)^{-q} s_2^{-q} \|v_\xi \varrho(\xi)\| \\ &= s_1^{-q} \|v_\xi \varrho(\xi)\|. \end{aligned}$$

Hence, for $t \in I$,

$$\left(\frac{\|v_\xi \varrho(gha_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}} > s_1.$$

The uniqueness of ξ_1 yields $\xi_1 = \xi$. Moreover, we can choose $g_1 = g$ for $t \in I$. As in the proof of Proposition 5.5, we see that we can choose $g_1 = g$ for all $t \in [1, r_0^S]$.

Proposition 5.5 shows that $g \in \xi NAMU$, say $g = \xi n_4 a_{r_1} m_1 u_1$ with $u_1 = \sigma(1, Z_1, X_1)\sigma$, and that

$$|X_1| < 2r_0^{-S/4} \quad \text{and} \quad |Z_1| < r_0^{-S/2}. \tag{10}$$

Suppose that $h = u_2 n_3 a_{r_2} m_2$ and set $h_2 := n_3 a_{r_2} m_2$. Then

$$\|v_\xi \varrho(gh)\| = \|v_\xi \varrho(gu_2 h_2)\| \le \|v_\xi \varrho(gu_2)\| \|\varrho(h_2)\| \le \|v_\xi \varrho(gu_2)\| \left(\frac{s_1}{s_2} \right)^{-q}$$

and

$$\begin{aligned}\|v_\xi \varrho(gu_2a^S)\| &= \|v_\xi \varrho(gha^S a^{-S} h_2^{-1} a^S)\| \\ &\le \|v_\xi \varrho(gha^S)\| \|\varrho(a^{-S}h_2^{-1}a^S)\| \le \|v_\xi \varrho(gha^S)\| \left(\frac{s_1}{s_2}\right)^{-q}.\end{aligned}$$

This yields

$$\begin{aligned}\left(\frac{\|v_\xi \varrho(gu_2a^S)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}} &\ge \frac{s_1}{s_2}\left(\frac{\|v_\xi \varrho(gha^S)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}} = \frac{s_1}{s_2}\operatorname{ht}(xha^S) \\ &> \frac{s_1}{s_2}\operatorname{ht}(xh) = \frac{s_1}{s_2}\left(\frac{\|v_\xi \varrho(gh)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}} \\ &\ge \left(\frac{s_1}{s_2}\right)^2\left(\frac{\|v_\xi \varrho(gu_2)\|}{\|v_\xi \varrho(\xi)\|}\right)^{-\frac{1}{q}}.\end{aligned}$$

Let $u_2 = \sigma(1, Z_2, X_2)\sigma$. Then

$$u_1u_2 = \sigma(1, Z_1 + Z_2 + \frac{1}{2}[X_1, X_2], X_1 + X_2)\sigma.$$

From (10) and $u_2 \in D_\kappa^U$ it follows that

$$|X_1 + X_2| \le |X_1| + |X_2| < 2 + \kappa.$$

Moreover, using triangle inequality and [Poh10, Lemma 2.12, Proposition 3.3] we find

$$|Z_1 + Z_2 + \frac{1}{2}[X_1, X_2]| \le |Z_1| + |Z_2| + \frac{1}{2}|X_1||X_2| < 1 + 2\kappa.$$

Thus, u_1u_2 is contained in the bounded set $D^U_{2+2\kappa}$. Note that this set only depends on κ. Then Lemma 6.2 gives

$$|X_1 + X_2| < c_1 r_0^{-S/4} \quad \text{and} \quad |Z_1 + Z_2 + \tfrac{1}{2}[X_1, X_2]| < c_2 r_0^{-S/2},$$

where the constants c_1, c_2 only depend on s_1, s_2 and κ. It follows that

$$|X_2| < c_1 r_0^{-S/4} + |X_1| < (c_1 + 2) r_0^{-S/4}$$

and

$$\begin{aligned}|Z_2| &\le |Z_1 + Z_2 + \tfrac{1}{2}[X_1, X_2]| + |Z_1| + \tfrac{1}{2}|X_1||X_2| \\ &\le c_2 r_0^{-S/2} + r_0^{-S/2} + (c_1 + 2) r_0^{-S/2}.\end{aligned}$$

This completes the proof. □

7. Estimate of metric entropy and main theorem

In this section we prove the main theorem. This section can be understood independently from the previous ones if the reader is willing to accept the following facts previously proven: The height level s_3 is chosen such that the connected parts of $\mathcal{X}_{>s_3}$ (thus, cuspidal ends of uniform "length") can be identified with $(\Gamma \cap P)\backslash C$, where C is the cylindrical set $C = \xi A_{s_3} NK$ at the cusp ξ of the considered end and P is the corresponding minimal parabolic subgroup in G. In particular, this means that connected parts of geodesic trajectories in $\mathcal{X}_{>s_3}$

can be identified with any representing geodesic trajectories in C. As a consequence we know (see Lemma 6.1) that (discretized) geodesic trajectories in $\mathcal{X}_{>s_3}$ which start to move out of the cusp actually descend to below height level s_3, and geodesics in $\mathcal{X}$ which move from one of these cuspidal ends to another one necessarily have to pass through the compact part $\mathcal{X}_{\leq s_3}$. Moreover, if two nearby points x, xh in $\mathcal{X}$ ($h \in G$) stay together near a cusp (meaning in the same connected component of $\mathcal{X}_{>s_3}$) for time t, then the unstable component of h is restricted (up to a multiplicative constant) by $t^{-1/2}$ in the direction of the long root and by $t^{-1/4}$ in the direction of the short root (see Proposition 6.3).

We first recall how to calculate the maximal metric entropy of T. Set $p_1 := \dim \mathfrak{g}_1$, $p_2 := \dim \mathfrak{g}_2$ and recall that $\widetilde{a} = a_{r_0}$.

Proposition 7.1. *The maximal metric entropy of T exists. It is*

$$h_m(T) = \left(\frac{p_1}{2} + p_2\right) \log r_0.$$

Proof. The statement follows from a combination of the proposition in Section 9.3 in [MT94] and Lemma 9.5 and Proposition 9.6 in [MT94]. If G is algebraic, a more accessible reference is [EL10, Theorem 7.6]. Note that

$$-\log \det \left(\mathrm{Ad}_a |_{\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}}\right) = \left(\frac{p_1}{2} + p_2\right) \log r_0.$$

□

Let $s > s_3$ and $L \in \mathbb{N}$. Let η be a finite partition of $\mathcal{X}$ of the form

$$\eta = \{\mathcal{X}_{>s}, \mathcal{X}_{>s_3} \cap \mathcal{X}_{\leq s}, P_1, \ldots, P_r\}$$

with $P_i \subseteq \mathcal{X}_{\leq s_3}$ for $i = 1, \ldots, r$. For any $P \in \eta_0^L = \bigvee_{j=0}^{L-1} T^{-j}\eta$ we define

$$V_P := \left\{ j \in \{0, \ldots, L-1\} \;\middle|\; T^j P \subseteq \mathcal{X}_{>s} \right\}. \tag{11}$$

Let λ be such that $r_0\lambda$ is less than the injectivity radius of $\mathcal{X}_{\leq s_3}$. We define

$$B_L := \bigcap_{j=0}^{L-1} \tilde{a}^j B_\lambda^G \tilde{a}^{-j}.$$

Any subset of $\mathcal{X}$ of the form

$$xB_L$$

with $x \in \mathcal{X}$ is called a *(forward) Bowen L-ball*. In Lemma 7.3 below we will estimate how many Bowen L-balls are needed to cover $P \in \eta_0^L$ for certain partitions η and $P \subseteq \mathcal{X}_{\leq s_3}$. Using the work of Brin and Katok [BK83] we will relate this estimate to the measure theoretic entropy.

From now on we assume that κ in (8) is chosen such that $D_\kappa \subseteq B_\lambda^G$. Then

$$\tilde{a}^{L-1} D_\kappa^U \tilde{a}^{-(L-1)} D_\kappa^{NAM} \subseteq B_L.$$

Therefore it suffices to estimate how many sets of the form $x\tilde{a}^{L-1} D_\kappa^U \tilde{a}^{-(L-1)} D_\kappa^{NAM}$ are needed to cover P to get an estimate on a sufficient number of Bowen L-balls. A set of the form

$$x\tilde{a}^{L-1} D_\kappa^U \tilde{a}^{-(L-1)} D_\kappa^{NAM}$$

will be called *L-box* with center x.

Lemma 7.2. *Let $s > s_3$. Then there exists $n_{max} \in \mathbb{N}$ such that whenever $x \in \mathcal{X}_{>s}$ satisfies $Tx, \ldots, T^n x \in \mathcal{X}_{\leq s} \cap \mathcal{X}_{>s_3}$, then $n \leq n_{max}$.*

Proof. Let $x \in \mathcal{X}_{>s}$ be as in the statement of the lemma. Lemma 6.1 and Proposition 5.5 show that there is a unique $\xi \in \Xi$ and some $g \in G$ such that $\Gamma g = x$ and

$$\mathrm{ht}(xa_t) = \left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}}$$

for $t \in [1, r_0^n]$. We suppose first that $g = \xi n a_r m \sigma$ for some $n \in N$, $r > 0$ and $m \in M$. Then

$$\mathrm{ht}(xa_t) = \frac{r}{t}$$

for $t \in [1, r_0^n]$. Therefore

$$s \geq \mathrm{ht}(Tx) = \frac{r}{r_0}.$$

This and

$$\mathrm{ht}(T^n x) = \frac{r}{r_0^n} > s_3$$

yield

$$n < \frac{\log \frac{r}{s_3}}{\log r_0} \leq \frac{\log \frac{s r_0}{s_3}}{\log r_0}.$$

Now we suppose that $g = \xi n a_r m \sigma (1, Z, X) \sigma$ for some $n, (1, Z, X) \in N$, $r > 0$ and $m \in M$. For $t \in [1, r_0^n]$ we have

$$\mathrm{ht}(xa_t) = r \cdot \frac{t^{-1}}{\left(t^{-1} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}.$$

Then $\mathrm{ht}(xa_t) > s_3$ is equivalent to

$$0 > \left(t^{-1} - \lambda_-\right)\left(t^{-1} - \lambda_+\right) \tag{12}$$

where

$$\lambda_\pm = -\frac{1}{2}\left(\frac{1}{2}|X|^2 - \frac{r}{s_3}\right) \pm \sqrt{\frac{1}{4}\left(\frac{1}{2}|X|^2 - \frac{r}{s_3}\right)^2 - \left(\frac{1}{16}|X|^4 + |Z|^2\right)}.$$

Since $\mathrm{ht}(x) > s_3$, (12) is satisfied at least for $t = 1$. Therefore, the roots $\lambda_\pm$ are real and

$$\lambda_+ > 1 > \lambda_-.$$

From $\lambda_+ > 1$ it follows that

$$\frac{1}{2}\left(\frac{r}{s_3} - \frac{1}{2}|X|^2\right) > 0.$$

In turn, $\lambda_- > 0$. Now $\mathrm{ht}(T^n x) > s_3$ implies

$$r_0^n < \lambda_-^{-1} = \frac{\frac{1}{2}\left(\frac{r}{s_3} - \frac{1}{2}|X|^2\right) + \sqrt{\frac{1}{4}\left(\frac{r}{s_3} - \frac{1}{2}|X|^2\right)^2 - \left(\frac{1}{16}|X|^4 + |Z|^2\right)}}{\frac{1}{16}|X|^4 + |Z|^2}.$$

From $s \geq \mathrm{ht}(Tx)$ it follows that

$$r \leq r_0 s \left[\left(r_0^{-1} + \frac{1}{4}|X|^2\right)^2 + |Z|^2 \right].$$

Therefore

$$\lambda_-^{-1} \leq \frac{r_0 s}{s_3} \cdot \frac{\left(r_0^{-1} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}{\frac{1}{16}|X|^4 + |Z|^2}$$
$$= \frac{s}{s_3} \cdot \frac{r_0^{-1} + \frac{1}{2}|X|^2}{\frac{1}{16}|X|^4 + |Z|^2} + \frac{r_0 s}{s_3}.$$

From $\mathrm{ht}(x) > \mathrm{ht}(Tx)$, a straightforward deduction yields

$$\frac{1}{16}|X|^4 + |Z|^2 > r_0^{-1}.$$

Hence,

$$\frac{r_0^{-1} + \frac{1}{2}|X|^2}{\frac{1}{16}|X|^4 + |Z|^2}$$

is bounded from above (independent of x), and so is λ_-^{-1}. This completes the proof. $\square$

In the following, for $s' > s > s_3$ we define various numbers which vary with s and s'.

Let ℓ denote the maximal number of T-steps between $\mathcal{X}_{>s}$ and $\mathcal{X}_{\leq s_3}$, that is,

$$\ell := \max\{k \in \mathbb{N} \mid \exists\, x \in \mathcal{X}_{>s} \colon Tx, \ldots, T^k x \in \mathcal{X}_{>s_3} \cap \mathcal{X}_{\leq s},\ T^{k+1}x \in \mathcal{X}_{\leq s_3}\}.$$

We note that this maximum exists by Lemma 7.2. It equals the maximal number of T-steps between $\mathcal{X}_{\leq s_3}$ and $\mathcal{X}_{>s}$ in the sense that

$$\ell := \max\{k \in \mathbb{N} \mid \exists\, x \in \mathcal{X}_{\leq s_3} \colon Tx, \ldots, T^k x \in \mathcal{X}_{>s_3} \cap \mathcal{X}_{\leq s},\ T^{k+1}x \in \mathcal{X}_{>s}\},$$

which follows since all sets of the form $\mathcal{X}_{\leq t}$ or $\mathcal{X}_{>t}$ are invariant under σ and since $\sigma \tilde{a} \sigma = \tilde{a}^{-1}$. We note that the maximal amount of time a trajectory can spend continuously within $\mathcal{X}_{>s_3} \cap \mathcal{X}_{\leq s}$ is then bounded by $2\ell + 5$ (corresponding to a trajectory that reaches about height s and then returns to $\mathcal{X}_{\leq s_3}$), i.e. that

$$\max\{k \in \mathbb{N} \mid \exists\, x \colon x, Tx, \ldots, T^k x \in \mathcal{X}_{>s_3} \cap \mathcal{X}_{\leq s}\} \leq 2\ell + 5$$

We define ℓ' in the same way using s' in place of s.

For the statement of the following lemma we remark that $\mathcal{X}_{\leq s}$ is compact by [Dan84, p. 27]. For shortness we use the notation

$$[m, n) := \{m, m+1, \ldots, n-1\}$$

for an interval of integer points with endpoints $m \leq n \in \mathbb{N}$.

An interval $k + [0, K) \subseteq [0, L)$ of a trajectory of a set $P \in \eta_0^L$ is said to be an *excursion into* $\mathcal{X}_{>s}$ (of length K) if

$$T^{k-1}P \subseteq X_{\leq s}, \quad T^k P, \ldots, T^{k+K-1}P \subseteq \mathcal{X}_{>s},$$
$$\text{and either } T^{k+K}P \subseteq \mathcal{X}_{\leq s} \text{ or } k + K = L.$$

Clearly, V_P is a union of intervals which are excursions into $\mathcal{X}_{>s}$.

Lemma 7.3. *Let $s' > s > s_3$ and define ℓ, ℓ' as above. Let $\lambda' > 0$ be such that $r_0\lambda'$ is less than the injectivity radius on $\mathcal{X}_{\leq s'}$ and that $B^G_{\lambda'} \subseteq D_\kappa$. Suppose that $\eta = \{\mathcal{X}_{>s}, \mathcal{X}_{>s_3} \cap \mathcal{X}_{\leq s}, P_1, \ldots, P_r\}$ is a finite partition of $\mathcal{X}$ such that*

$\operatorname{diam} T^j(P_i) \le \lambda'$ *for each* $j = 1, \dots, 2\ell' + 5$ *and* $i = 1, \dots, r$. *Then for each* $L \in \mathbb{N}$ *and* $P \in \eta_0^L$ *with* $P \subseteq \mathcal{X}_{\le s_3}$ *the set* P *can be covered by*

$$c^m e^{h_m(T)\ell m} e^{\frac{1}{2}h_m(T)|V_P|}$$

Bowen L*-balls. Here the constant* c *only depends on the group* G *and the action* T*, and* m *is the number of excursions of* P *into* $\mathcal{X}_{>s'}$*.*

We note that while the partition is (in a strong way) adapted to the heights s, s', our definition of Bowen L-ball does not depend on s, s' – we only insist that the radius used in the definition is less than the injectivity radius on $\mathcal{X}_{\le s_3}$.

Proof. Let $P \in \eta_0^L$ with $P \subseteq \mathcal{X}_{\le s_3}$. Let $V := V_P$. We decompose V into a disjoint union

$$V = \bigcup_{i=1}^{m_1} V_i = [k_1, k_1 + K_1) \cup \ldots \cup [k_{m_1}, k_{m_1} + K_{m_1}) \tag{13}$$

where each V_i is an excursion into $\mathcal{X}_{>s}$, i.e. a maximal subset of V of the form $[k_i, k_i + K_i)$. We may suppose that $k_1 < k_2 < \cdots < k_{m_1}$.

We note that each excursion V_i into $\mathcal{X}_{>s}$ is contained in an excursion $\tilde{V}_i \subseteq [k_i - \ell, k_i + K_i + \ell)$ into $\mathcal{X}_{>s_3}$. We define $\tilde{V} = \bigcup_{i=1}^{m_1} \tilde{V}_i$. Finally we define $\bar{V}$ as the union of all excursions $\tilde{V}_i$ that contain excursions into $\mathcal{X}_{>s'}$. Below we wish to treat these intervals by using the assumption of the lemma only.

Analogously, we decompose $W := [0, L) \smallsetminus \bar{V}$ into a disjoint union

$$W = \bigcup_{j=1}^{m+1} W_j$$

where each W_j is a maximal subset of W of the form $[l_j, l_j + L_j)$ with $0 = l_1 < l_2 < \cdots < l_{m+1}$. The set W_{m+1} might be empty. Also let us define $\bar{V}_j = [n_j, n_j + h_j)$ for $j = 1, \dots, m$ as those intervals from the list $\tilde{V}_i$ for $i = 1, \dots, m_1$ that contain excursions into $\mathcal{X}_{>s'}$ and assume $n_1 < n_2 < \ldots < n_m$.

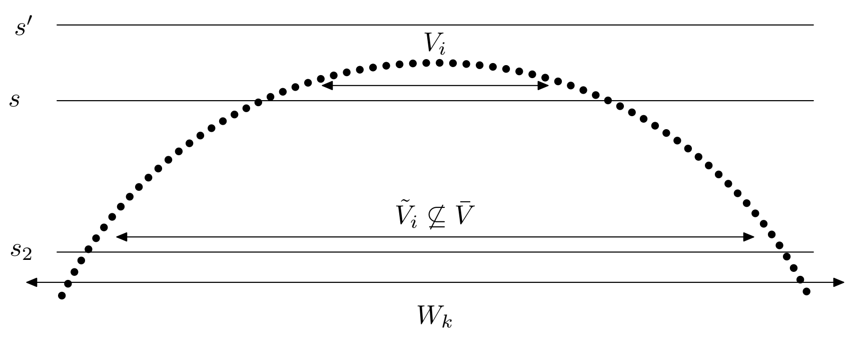

FIGURE 1. One possible situation

The step corresponding to W_j*:* Note first that even though an interval $W_j = [l_j, l_j + L_j)$ may contain one or many excursions into $\mathcal{X}_{>s}$, it does not contain

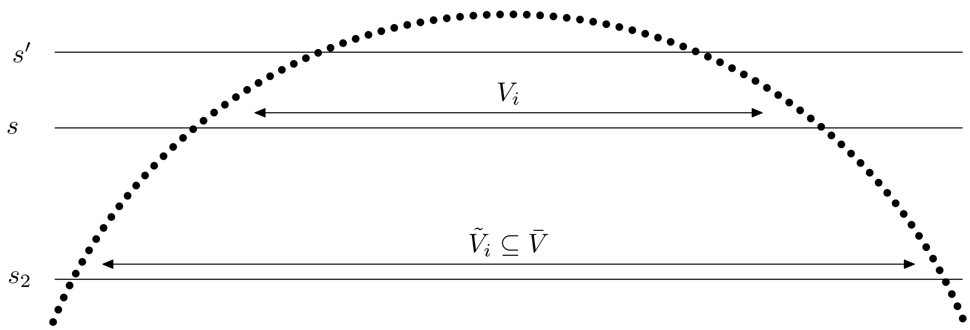

FIGURE 2. The other possible situation

an excursion into $\mathcal{X}_{>s'}$. Each of the excursions into $\mathcal{X}_{>s}$ is of length at most $2\ell' + 5$.

Suppose now inductively that we have already found (at most)

$$c^{j-1} e^{\frac{1}{2} h_m(T) |\bar{V}_1 \cup \cdots \cup \bar{V}_{j-1}|} \tag{14}$$

$l_j + 1$-boxes with center in P whose union contains the given element $P \in \eta_0^L$. Here c is an absolute constant (depending only on G, T, s_3 but not on s or s'), which we determine below. If $j = 1$, and hence $l_j = 0$, this is a trivial claim as the diameter of P is assumed to be quite small.

The choice of λ', ℓ' and η implies that any element $Q \in \eta$ with $Q \subseteq \mathcal{X}_{\leq s_3}$ is contained in

$$w \bigcap_{i=0}^{k} \tilde{a}^i B^G_{\lambda'} \tilde{a}^{-i} \tag{15}$$

for $k \leq 2\ell' + 5$ and for any $w \in Q$ as long as $Q, T(Q), \ldots, T^k(Q) \subseteq \mathcal{X}_{\leq s'}$. A simple induction shows that $Q \in \eta_0^{L'}$ belongs to the "small" Bowen L'-ball (15) with $k = L' - 1$ if it is known that $Q, T(Q), \ldots, T^{L'-1}(Q) \subseteq \mathcal{X}_{\leq s'}$. This applies to $Q = T^{l_j}(P)$ and $L' = L_j$ and shows that $T^{l_j}(P)$ is contained in

$$w \bigcap_{i=0}^{L_j - 1} \tilde{a}^i B^G_{\lambda'} \tilde{a}^{-i} \tag{16}$$

for any $w \in T^{l_j}(P)$. Set now $B := \tilde{a}^{l_j} D^U_\kappa \tilde{a}^{-l_j} D^{NAM}_\kappa$ and let xB be one of the sets used in (14). Set $w := x\tilde{a}^{l_j}$. From $x\tilde{a}^{l_j} \in \mathcal{X}_{\leq s_3}$ and the choice of λ' it follows

that

$$\begin{aligned}
&P \cap xB \\
&\subseteq \left((x\tilde{a}^{l_j}) \bigcap_{i=0}^{L_j-1} \tilde{a}^i B^G_{\lambda'} \tilde{a}^{-i} \cap (x\tilde{a}^{l_j}) D^U_\kappa \left(\tilde{a}^{-l_j} D^{NAM}_\kappa \tilde{a}^{l_j} \right) \right) \tilde{a}^{-l_j} \\
&= x \Big(\bigcap_{i=0}^{L_j-1} \tilde{a}^{l_j+i} B^G_{\lambda'} \tilde{a}^{-(l_j+i)} \cap B \Big) \\
&\subseteq x \left(\tilde{a}^{l_j+L_j-1} D^U_\kappa \tilde{a}^{-(l_j+L_j-1)} \tilde{a}^{l_j} D^{NAM}_\kappa \tilde{a}^{-l_j} \cap \tilde{a}^{l_j} D^U_\kappa \tilde{a}^{-l_j} D^{NAM}_\kappa \right) \\
&= x\tilde{a}^{l_j+L_j-1} D^U_\kappa \tilde{a}^{-(l_j+L_j-1)} D^{NAM}_\kappa .
\end{aligned}$$

Note that $n_j = l_j + L_j$, which is the starting point of $\bar{V}_j$ for $j \leq m$. Thus, P is covered by at most

$$c^{j-1} e^{\frac{1}{2} h_m(T) |\bar{V}_1 \cup \cdots \cup \bar{V}_{j-1}|} \tag{17}$$

many n_j-boxes with center in P.

The step corresponding to $\bar{V}_j$: Suppose now inductively that we have already found (17)-many n_j-boxes with center in P that cover P. Set

$$B := \tilde{a}^{n_j-1} D^U_\kappa \tilde{a}^{-(n_j-1)} D^{NAM}_\kappa$$

and let xB be any n_j-box used in this covering. Define

$$E := \left\{ y \in xB \;\middle|\; T^{n_j-1} y \in \mathcal{X}_{\leq s_3}, T^{n_j} y, \ldots, T^{n_j+h_j-1} y \in \mathcal{X}_{> s_3} \right\}.$$

Then $P \cap xB \subseteq E$. If $y \in E$, then there exists $g \in B$ with $y = xg$. Say $g = \tilde{a}^{n_j-1} \sigma(1, Z, X) \sigma \tilde{a}^{-(n_j-1)} n a_r m$. Set $x' := T^{n_j-1} x$ and $y' := T^{n_j-1} y$. Then $y' = x'h$ with

$$h := \tilde{a}^{-(n_j-1)} g \tilde{a}^{n_j-1} = \sigma(1, Z, X) \sigma n' a_r m$$

and $n' := \tilde{a}^{-(n_j-1)} n \tilde{a}^{n_j-1}$. Recall that

$$\tilde{a}^{-(n_j-1)} D^{NAM}_\kappa \tilde{a}^{n_j-1} \subseteq D^{NAM}_\kappa .$$

Thus $h \in D^U_\kappa D^{NAM}_\kappa$. Moreover

$$\begin{aligned}
Tx', \ldots, T^{h_j} x' &\in \mathcal{X}_{> s_3}, \quad x' \in \mathcal{X}_{\leq s_3} \cap \mathcal{X}_{> s_2}, \\
Ty', \ldots, T^{h_j} y' &\in \mathcal{X}_{> s_3}, \quad y' \in \mathcal{X}_{\leq s_3} \cap \mathcal{X}_{> s_2}.
\end{aligned}$$

Applying Proposition 6.3 to x', y' shows that

$$|Z| \leq c_4 r_0^{-h_j/2} \quad \text{and} \quad |X| \leq c_3 r_0^{-h_j/4}.$$

Now

$$\left\{ \sigma(1, Z, X)\sigma \;\middle|\; |Z| \leq c_4 r_0^{-h_j/2},\ |X| \leq c_3 r_0^{-h_j/4} \right\}$$

can be covered with

$$\left(2 \frac{c_3}{\kappa} r_0^{h_j/4} \right)^{p_1} \cdot \left(2 \frac{c_4}{\kappa} r_0^{h_j/2} \right)^{p_2}$$

U-translates of $\tilde{a}^{h_j} D^U_\kappa \tilde{a}^{-h_j}$. Further, if

$$Q := P \cap z\tilde{a}^k D^U_\kappa \tilde{a}^{-k} D^{NAM}_\kappa \neq \emptyset$$

for some $k \in \mathbb{N}$ and some $z \in \mathcal{X}$, then Q is covered by a finite number of sets of the form $y\tilde{a}^k D^U_\kappa \tilde{a}^{-k} D^{NAM}_\kappa$ with $y \in P$. The number of necessary sets is

bounded by a constant, say c_5, independent of j, P and z. Hence we need at most

$$c_5 2^{p_1+p_2}\left(\frac{c_3}{\kappa}\right)^{p_1}\left(\frac{c_4}{\kappa}\right)^{p_2} e^{\frac{1}{2}h_m(T)h_j}$$

(n_j+h_j)-boxes with center in P to cover $P\cap xB$. Combining this with the inductive assumption we get that P is contained in at most

$$c^j e^{\frac{1}{2}h_m(T)|\bar{V}_1\cup\cdots\cup\bar{V}_j|}$$

many $l_{j+1}+1$-boxes (where $l_{j+1}=n_j+h_j$). This concludes the induction.

Finally note that either (14) for $j=m$ (if $W_{m+1}=\emptyset$) or (17) for $j=m+1$ gives the conclusion of the proposition since $|\bar{V}_k|$ is at most by 2ℓ bigger than the corresponding $|V_i|$. □

Proposition 7.4. *For all $s>s_3$ there exists a finite partition $\eta=\{\mathcal{X}_{>s},\mathcal{X}_{>s_3}\cap\mathcal{X}_{\le s},P_1,\ldots,P_r\}$ of $\mathcal{X}$ such that for each T-invariant probability measure μ on $\mathcal{X}$ we have*

$$h_\mu(T)\le h_\mu(T,\eta)+\tfrac{1}{s}+\tfrac{1}{2}h_m(T)\big(1-\mu(\mathcal{X}_{\le s})\big).$$

Proof. Using the ergodic decomposition of invariant measures we may restrict to ergodic and invariant measures. Also note that every T-orbit visits $\mathcal{X}_{\le s_3}$ (Lemma 6.1(v), which also holds for T^{-1} in place of T), so that we must have $\delta_1:=\mu(\mathcal{X}_{\le s_3})>0$. Let $s'>s$. Define ℓ as above and let

$$\ell'':=\min\left\{k\in\mathbb{N}\,\middle|\,\exists\, x\in\mathcal{X}_{\le s_3}\colon T^{k+1}x\in\mathcal{X}_{>s'}\right\}.$$

Let $\eta=\{\mathcal{X}_{>s},\mathcal{X}_{>s_3}\cap\mathcal{X}_{\le s},P_1,\ldots,P_r\}$ be as in Lemma 7.3.

We will show the proposition using [BK83], more precisely in the form of Lemma B.2 in [ELMV]: There it is shown that the entropy of μ is the limit as $\kappa\to 0$ of

$$\limsup_{L\to\infty}\frac{\log N_\kappa(\delta,L)}{L},$$

where $N_\kappa(\delta,L)$ is the minimal number of Bowen L-balls, with κ being the radius used in the definition, that are needed to cover at least a set of μ-measure δ for some $\delta>0$. However, there is one important difference between our definition of Bowen L-balls and that of [BK83]. In the latter, one takes the intersections of pre-images of κ-balls within $\mathcal{X}$. In our definition of Bowen L-ball we took the intersection in the group, which results in general in a smaller set (namely in those cases where the orbit ventures near the cusp). As we are seeking an upper bound of entropy, we may use [BK83] also together with our definition of Bowen L-balls. This has the advantage that we do not have to take the limit as $\kappa\to 0$. Within the group a bounded number of translates of Bowen L-balls defined by $\kappa>0$ can be used to cover a Bowen L-ball defined by $\kappa'>0$, and as $L\to\infty$ this difference becomes unimportant.

By ergodicity $\bigcup_{j=0}^{\infty}T^{-j}\mathcal{X}_{\le s_3}$ has measure one, and so there exists M with

$$\mu\bigg(\bigcup_{j=0}^{M-1}T^{-j}\mathcal{X}_{\le s_3}\bigg)>1-\frac{\delta_1}{2}.$$

The intersection of the preimage of this set under $T^{L'}$ with $\mathcal{X}_{\leq s_3}$ has measure at least $\delta_1/2$. It follows that there are infinitely many L (of the form $L'+j$ for some $j \in [0, M)$) for which

$$\mu\big(\mathcal{X}_{\leq s_3} \cap T^{-L}\mathcal{X}_{\leq s_3}\big) > \frac{\delta_1}{2M}.$$

We now proceed making $Y_L = \mathcal{X}_{\leq s_3} \cap T^{-L}\mathcal{X}_{\leq s_3}$ smaller, taking care that the resulting sets have measures that do not approach zero, and obtaining more information on the smaller sets.

Since μ is ergodic the number $h_\mu(T,\eta)$ has the following interpretation: for every $\epsilon > 0$ and every sufficiently large L there exists a set $Z_{L,\epsilon}$ such that its measure is bigger than $1-\epsilon$ and $Z_{L,\epsilon}$ can be covered by at most $e^{(h_\mu(T,\eta)+\epsilon)L}$ many elements of η_0^L. We choose $\epsilon = \min(\frac{1}{3s}, \frac{\delta_1}{4M})$, and take the intersection $Y'_L = Y_L \cap Z_{L,\epsilon}$. We now know that $\mu(Y'_L) > \frac{\delta_1}{4M}$ and that Y'_L can be covered by at most $e^{(h_\mu(T,\eta)+\frac{1}{3s})L}$ many elements of η_0^L.

Finally, we may make Y'_L again a bit smaller to ensure that the ergodic averages for the characteristic function $\mathcal{X}_{>s}$ are correct up to an error of $(3sh_m(T))^{-1}$ and for sufficiently large L. More precisely there exists a subset $Y''_L \subset Y'_L$ (obtained by intersecting Y'_L with a set of near full measure) with $\mu(Y''_L) > \delta = \frac{\delta_1}{5M}$ and some L_0 such that for all $L > L_0$ and all $x \in Y''_L$ we have

$$\left| \frac{1}{L}\sum_{i=0}^{L-1} \chi_{\mathcal{X}_{>s}}(T^i x) - \mu(\mathcal{X}_{>s}) \right| < \frac{1}{3sh_m(T)}.$$

Now apply Lemma 7.3 to each of the partition elements of η_0^L obtained earlier. Notice that for each of the partition elements we have $m \leq \frac{L}{\ell''}$. Finally, notice that the above ergodic sum is constant on each $P \in \eta_0^L$. For those P that intersect Y''_L we then have $|V_P| < (\mu(\mathcal{X}_{>s}) + \frac{1}{3sh_m(T)})L$. Using this together with the above it follows that Y''_L can be covered by N_L-many Bowen L-balls with

$$N_L \leq e^{(h_\mu(T,\eta)+\frac{1}{3s})L}\big(ce^{h_m(T)\ell}\big)^{\frac{L}{\ell''}} e^{\frac{1}{2}h_m(T)\mu(\mathcal{X}_{>s})L + \frac{1}{6s}L}.$$

Choose s' so big such that $\frac{\ell}{\ell''} < \frac{1}{s}$. This implies the proposition. $\square$

Theorem 7.5. *Let (μ_j) be a sequence of T-invariant probability measures on $\mathcal{X}$ which converges to the measure ν. Then*

$$\nu(\mathcal{X})h_{\frac{\nu}{\nu(\mathcal{X})}}(T) + \tfrac{1}{2}h_m(T)\big(1-\nu(\mathcal{X})\big) \geq \limsup_{j\to\infty} h_{\mu_j}(T),$$

where it does not matter how we interprete $h_{\frac{\nu}{\nu(\mathcal{X})}}(T)$ if $\nu(\mathcal{X}) = 0$.

Proof. Pick $s > s_3$ such that $\nu(\partial\mathcal{X}_{\leq s}) = 0$ (this holds for Lebesgue almost all s). Let $\eta = \{\mathcal{X}_{>s}, \mathcal{X}_{>s_3} \cap \mathcal{X}_{\leq s}, P_1, \ldots, P_r\}$ be a partition of $\mathcal{X}$ as in Proposition 7.4 such that $\nu(\partial P_j) = 0$ for $j = 1, \ldots, r$. Let $\varepsilon > 0$. For $m \in \mathbb{N}$ set

$$\Xi_m := \bigvee_{k=0}^{m-1} T^{-k}\eta.$$

Suppose now that $\nu(\mathcal{X}) > 0$. Fix $m \in \mathbb{N}$ such that

$$h_{\frac{\nu}{\nu(\mathcal{X})}}(T) + \varepsilon > \frac{1}{m}H_{\frac{\nu}{\nu(\mathcal{X})}}(\Xi_m)$$

and

$$\frac{2e^{-1}}{m} < \frac{\varepsilon}{2} \quad \text{and} \quad -\frac{1}{m}\log\nu(\mathcal{X}) < \varepsilon.$$

Then

$$\nu(\mathcal{X})h_{\frac{\nu}{\nu(\mathcal{X})}}(T) + 2\varepsilon > -\frac{1}{m}\sum_{P\in\Xi_m}\nu(P)\log\nu(P).$$

Note that this holds trivially if $\nu(\mathcal{X}) = 0$.

Let

$$Q := \bigcap_{k=0}^{m-1} T^{-k}\mathcal{X}_{>s}.$$

Since

$$\sum_{P\in\Xi_m\setminus\{Q\}} \mu_j(P)\log\mu_j(P) \overset{j\to\infty}{\longrightarrow} \sum_{P\in\Xi_m\setminus\{Q\}}\nu(P)\log\nu(P),$$

we find $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have

$$\begin{aligned}
&\left|-\frac{1}{m}\sum_{P\in\Xi_m}\nu(P)\log\nu(P) - \frac{1}{m}H_{\mu_j}(\Xi_m)\right| \\
&\quad\leq \frac{1}{m}\left|\sum_{P\in\Xi_m\setminus\{Q\}}\big(\mu_j(P)\log\mu_j(P) - \nu(P)\log\nu(P)\big)\right| \\
&\qquad\qquad + \frac{1}{m}\big|\mu_j(Q)\log\mu_j(Q) - \nu(Q)\log\nu(Q)\big| \\
&\quad\leq \frac{\varepsilon}{2} + \frac{2e^{-1}}{m} < \varepsilon.
\end{aligned}$$

This and Proposition 7.4 yield

$$\begin{aligned}
\nu(\mathcal{X})h_{\frac{\nu}{\nu(\mathcal{X})}}(T) + 3\varepsilon &> \frac{1}{m}H_{\mu_j}(\Xi_m) \geq h_{\mu_j}(T,\eta) \\
&> h_{\mu_j}(T) - \tfrac{1}{s} - \tfrac{1}{2}h_m(T)\cdot\big(1-\mu_j(\mathcal{X}_{\leq s})\big).
\end{aligned}$$

Hence

$$\nu(\mathcal{X})h_{\frac{\nu}{\nu(\mathcal{X})}}(T) + \tfrac{1}{2}h_m(T)\cdot\big(1-\nu(\mathcal{X}_{\leq s})\big) + 3\varepsilon + \tfrac{1}{s} \geq \limsup_{j\to\infty} h_{\mu_j}(T).$$

Letting ε tend to 0 and s tend to infinity, it follows

$$\nu(\mathcal{X})h_{\frac{\nu}{\nu(\mathcal{X})}}(T) + \tfrac{1}{2}h_m(T)\cdot\big(1-\nu(\mathcal{X})\big) \geq \limsup_{j\to\infty} h_{\mu_j}(T).$$

□

An immediate consequence of Theorem 7.5 is the following result about escape of mass.

Corollary 7.6. *Let $(\mu_j)_{j\in\mathbb{N}}$ be a sequence of T-invariant probability measures on $\mathcal{X}$ such that $\liminf_{n\to\infty} h_{\mu_j}(T) \geq c$. Let ν be any weak* accumulation point of (μ_j). Then*

$$\nu(\mathcal{X}) \geq \frac{2c}{h_m(T)} - 1.$$

Moreover, if

$$\nu(\mathcal{X}) = \frac{2c}{h_m(T)} - 1 > 0,$$

then $h_{\frac{\nu}{\nu(\mathcal{X})}}(T) = h_m(T)$.

8. Hausdorff dimension of orbits missing a fixed open subset

In this section we provide an application of Theorem 7.5 and the methods for its proof to answer a question by Barak Weiss about the Hausdorff dimension of the set of all orbits which miss a fixed open subset of $\mathcal{X}$. We note that for a compact quotient this is a simple corollary of semi-continuity of entropy and uniqueness of the measure of maximal entropy. In the presense of cusps the methods of this paper become relevant. We also note that related results have been obtained by Shi [Shi] but to our knowledge these do not provide the following results as corollaries.

Let $\mathcal{O} \subseteq \mathcal{X}$ be a non-empty open subset. Let $\mathcal{E}$ denote the set of points in $\mathcal{X}$ whose forward-A-orbits do not intersect $\mathcal{O}$, that is

$$\mathcal{E} := \{x \in \mathcal{X} \mid \forall\, t \geq 0\colon xa_t \notin \mathcal{O}\}.$$

In the following we will show that $\mathcal{E}$ cannot have full Hausdorff dimension.

Theorem 8.1. *We have*

$$\dim_H \mathcal{E} < \dim_H \mathcal{X}.$$

Instead of Theorem 8.1 we will prove a (stronger) discretized version. To that end let

$$T\colon \mathcal{X} \to \mathcal{X}, \quad x \mapsto xa_e$$

denote the time-one (discrete) geodesic flow. Note that then the maximal metric entropy is

$$h_m(T) = \frac{p_1}{2} + p_2.$$

We consider the set

$$\mathcal{E}' := \{x \in \mathcal{X} \mid \forall\, n \in \mathbb{N}_0\colon T^n x \notin \mathcal{O}\}.$$

Then Theorem 8.1 is implied by

Theorem 8.2. *We have*

$$\dim_H \mathcal{E}' < \dim_H \mathcal{X}.$$

of which we will provide a proof in the following. The strategy for its proof is as follows: We cover $\mathcal{E}'$ by countably many (small) open sets, say by $B_n, n \in \mathbb{N}$, and estimate the Hausdorff dimension of each of the sets

$$\mathcal{W}_n := \mathcal{E}' \cap B_n.$$

By countably stability of Hausdorff dimension we have

$$\dim_H \mathcal{E}' = \sup\{\dim_H \mathcal{W}_n \mid n \in \mathbb{N}\}.$$

Thus, we have to show that

$$\dim_H \mathcal{W}_n < \dim_H \mathcal{X} - \varepsilon_0$$

for some $\varepsilon_0 > 0$ not depending on $n \in \mathbb{N}$. To seek a contradiction we assume that for (a sequence of) arbitrarily small $\varepsilon_0 > 0$ we find a set $\mathcal{W} = \mathcal{W}(\varepsilon_0)$ among the sets $\mathcal{W}_n$ such that

$$d := \dim_H \mathcal{W} = \dim_H \mathcal{X} - \varepsilon_0,$$

For any $\delta \in (0, d)$, Frostman's Lemma assures the existence of a probability measure μ on $\mathcal{W}$ such that

$$\mu(xB_r^G) \leq cr^{d-\delta} \tag{18}$$

for any $x \in \mathcal{X}$ and any $r > 0$, with a constant c only depending on μ and $\mathcal{W}$. Then we will estimate the number of L-boxes (or Bowen L-balls) needed at most to cover $\mathcal{W}$ as well as the μ-mass of an L-box. Bounding the μ-mass of $\mathcal{W}$ (which is 1) via these L-boxes will result in a contradiction.

We start by choosing a good value for δ. Let $\mathcal{M}_1(\mathcal{X})^T$ denote the space of T-invariant probability measures on $\mathcal{X}$.

Lemma 8.3. *There exists $\delta_0 > 0$ such that*

$$\sup\{h_\nu(T) \mid \nu \in \mathcal{M}_1(\mathcal{X})^T,\ \operatorname{supp}\nu \subseteq \mathcal{E}'\} = h_m(T) - \delta_0.$$

Proof. To seek a contradiction assume that there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers with $\lim \varepsilon_n = 0$ and a sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_1(\mathcal{X})^T$ with $\operatorname{supp}\nu_n \subseteq \mathcal{E}'$ for each $n \in \mathbb{N}$ such that

$$h_{\nu_n}(T) = h_m(T) - \varepsilon_n.$$

Let ν be any weak* limit point of (ν_n). Then $\operatorname{supp}\nu \subseteq \mathcal{E}'$. Since $\liminf h_{\nu_n}(T) = h_m(T)$, Corollary 7.6 yields $\nu(\mathcal{X}) = 1$ and $h_\nu(T) = h_m(T)$. Thus, ν is the Haar measure on $\mathcal{X}$ and hence $\operatorname{supp}\nu = \mathcal{X}$. This is a contradiction. $\square$

As exponent δ in (18) we choose

$$\delta := \frac{\delta_0}{4}.$$

We pick a weak* limit ν of

$$\frac{1}{L}\sum_{j=0}^{L-1} T_*^j \mu \qquad \text{as } L \to \infty.$$

Then ν is T-invariant and $\nu(\mathcal{O}) = 0$. For sufficiently small ε_0, we see that $\nu(\mathcal{X}) > 0$ (see Proposition 8.7 below). Then Lemma 8.3 shows

$$h_{\frac{\nu}{\nu(\mathcal{X})}}(T) \leq h_m(T) - \delta_0.$$

Our next goal is to derive a lower bound for $\nu(\mathcal{X})$.

Lemma 8.4. *For any $r > 0$, any $x \in \mathcal{X}$ and any $L \in \mathbb{N}$ we have*

$$\mu(xa_e^L D_r^U a_e^{-L} D_r^{NAM}) \leq cr^{d-\frac{\delta_0}{4}} e^{\left(-h_m(T)+\varepsilon_0+\frac{\delta_0}{4}\right)L},$$

where the constant c is possibly slightly bigger than in (18).

Proof. Let $r > 0$ and $x \in \mathcal{X}$. Recall the definition of D_r from (8). The ball xB_r^G is covered by finitely many boxes yD_r ($y \in \mathcal{X}$), where this number only depends on the geometry of G and the norm chosen for the definition of B_r^G. The constant in the subhomogeneity property of the Frostman measure μ may be modified with this finite number. Without loss of generality we may assume that $B_r^G = D_r$. We have

$$a_e^L D_r^U a_e^{-L} = \left\{ \sigma(1, Z, X)\sigma \,\middle|\, |Z| < re^{-L},\ |X| < re^{-\frac{1}{2}L} \right\}.$$

Thus, $a_e^L D_r^U a_e^{-L}$ is covered by at most $2^{\dim U} e^{\frac{p_1}{2}L}$ translates of $D^U_{re^{-L}}$. Further, $D^U_{re^{-L}} D_r^{NAM}$ is covered by at most $2^{\dim(NAM)} e^{\dim(NAM)\cdot L}$ translates of $D^U_{re^{-L}} D^{NAM}_{re^{-L}}$. In total, $a_e^L D_r^U a_e^{-L} D_r^{NAM}$ is covered by at most

$$2^{\dim G} e^{\left(\frac{p_1}{2} + \dim NAM\right)L}$$

translates of $D_{re^{-L}}$. Thus

$$\begin{aligned} \mu(xa_e^L D_r^U a_e^{-L} D_r^{NAM}) &\le 2^{\dim G} e^{\left(\frac{p_1}{2} + \dim NAM\right)L} \mu(xD_{re^{-L}}) \\ &< cr^{d - \frac{\delta_0}{4}} e^{\left(\frac{p_1}{2} + \dim NAM - d + \frac{\delta_0}{4}\right)L} \\ &\le cr^{d - \frac{\delta_0}{4}} e^{(-h_m(T) + \varepsilon_0 + \frac{\delta_0}{4})L}. \end{aligned}$$

Here we modified the constant of the Frostman measure μ by $2^{\dim G}$. This completes the proof. $\square$

For $L \in \mathbb{N}$, a subset $V \subseteq [0, L-1]$ and $s > 0$ we set

$$Z_{s,L}(V) := \left\{ x \in \mathcal{W} \cap \mathcal{X}_{\le s} \,\middle|\, \forall j \in [0, L-1]\colon \left(T^j x \in \mathcal{X}_{>s} \Leftrightarrow j \in V\right) \right\}.$$

Lemma 8.5. *Let $s > s_3$ and*

$$L \ge 2\log\left(\frac{s}{s_3}\right) + 1.$$

Then there are at most

$$\exp\left(\frac{4\log\left(2\log\left(\frac{s}{s_3}\right) + 2\right)}{\log\left(\frac{s}{s_3}\right)} L \right)$$

subsets $V \subseteq [0, L-1]$ for which $Z_{s,L}(V)$ is nonempty.

Proof. Let $s > s_3$. For a subset $V \subseteq [0, L-1]$ we set

$$Q_{s,V} := \left\{ x \in \mathcal{X} \,\middle|\, \forall j \in [0, L-1]\colon \left(T^j x \in \mathcal{X}_{>s} \Leftrightarrow j \in V\right) \right\}.$$

In the following we estimate the number of subsets $V \subseteq [0, L-1]$ for which $Q_{s,V} \neq \emptyset$.

Let V be any subset of $[0, L-1]$ and decompose V as in (13). We will show that $Q_{s,V}$ being nonempty implies that there is a uniform nontrivial minimal distance between $k_n + K_n$ and k_{n+1}. The existence of this distance yields restrictions on those V for which $Q_{s,V} \neq \emptyset$.

At first suppose that we have $x \in \mathcal{X}$ with $\mathrm{ht}(x) \le s_3$ and $\mathrm{ht}(T^j x) > s$ for some $j \in \mathbb{N}$. By Lemma 6.1(iii) we may suppose $\mathrm{ht}(xa_t) > 2s_1$ for all $t \in [1, e^j]$. We

aim to prove a nontrivial lower bound on j. By Proposition 5.5 there exists a unique cusp $\xi \in \Xi$ and an element $g \in G$ with $x = \Gamma g$ such that

$$\mathrm{ht}(xa_t) = \mathrm{ht}_\xi(xa_t) = \left(\frac{\|v_\xi \varrho(ga_t)\|}{\|v_\xi \varrho(\xi)\|} \right)^{-\frac{1}{q}}$$

for all $t \in [1, e^j]$. Proposition 5.3 implies $g \in \xi NAMU$, say $g = \xi n a_r m u$ with $u = \sigma(1, Z, X)\sigma$. Lemma 5.2 shows

$$\mathrm{ht}_\xi(xa_t) = r \cdot \frac{\frac{1}{t}}{\left(\frac{1}{t} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}$$

for all $t \in [1, e^j]$. In particular,

$$s_3 > \mathrm{ht}_\xi(x) = \frac{r}{\left(1 + \frac{1}{4}|X|^2\right)^2 + |Z|^2}$$

and

$$\mathrm{ht}(xa_{e^j}) = r \cdot \frac{e^{-j}}{\left(e^{-j} + \frac{1}{4}|X|^2\right)^2 + |Z|^2} > s.$$

Therefore

$$\frac{\left(1 + \frac{1}{4}|X|^2\right)^2 + |Z|^2}{\left(e^{-j} + \frac{1}{4}|X|^2\right)^2 + |Z|^2} \cdot e^{-j} > \frac{s}{s_3}.$$

From

$$e^{2j} \geq \frac{\left(1 + \frac{1}{4}|X|^2\right)^2 + |Z|^2}{\left(e^{-j} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}$$

it follows that

$$j > \log\left(\frac{s}{s_3}\right).$$

Suppose now that we have $x \in \mathcal{X}$ with $\mathrm{ht}(x) > s$ and $\mathrm{ht}(T^j x) \leq s_3$ for some $j \in \mathbb{N}$. Invoking Lemma 6.1(iv), we can deduce as before that

$$j > \log\left(\frac{s}{s_3}\right).$$

We set

$$j_0 := \left\lceil \log\left(\frac{s}{s_3}\right) \right\rceil.$$

Lemma 6.1(v) implies that

$$k_n + K_n + 2j_0 \leq k_{n+1}$$

for $n = 1, \ldots, m_1 - 1$. Let

$$Q_{s,L} := \bigvee_{j=0}^{L-1} T^{-j}\{\mathcal{X}_{\leq s}, \mathcal{X}_{>s}\}.$$

If $L = 2j_0 - 1$, the cardinality of $Q_{s,L}$ is

$$1 + \binom{2j_0}{2},$$

which we estimate from above (note that $j_0 \geq 1$) with

$$(2j_0)^2.$$

For an arbitrary L the set $[0, L-1]$ is covered by the disjoint union

$$[0, L-1] \subseteq \bigcup_{h=0}^{k_L} h \cdot (2j_0 - 1) + [0, 2j_0 - 2]$$

with

$$k_L := \left\lceil \frac{L}{2j_0 - 1} \right\rceil .$$

For each $h \in \{0, \dots, k_L\}$, the cardinality of

$$\{V \cap (h \cdot (2j_0 - 1) + [0, 2j_0 - 2]) \mid V \subseteq [0, L-1],\ Q_{s,V} \neq \emptyset\}$$

is at most $(2j_0)^2$. Therefore, there are at most

$$(2j_0)^{2k_L}$$

subsets $V \subseteq [0, L-1]$ with $Q_{s,V} \neq \emptyset$. Hence the cardinality of $Q_{s,L}$ is bounded from above by

$$\exp\left(2k_L \cdot \log(2j_0)\right).$$

Using

$$k_L \leq \frac{L}{2j_0 - 1} + 1$$

and

$$\frac{\log\left(2\left\lceil \log\left(\frac{s}{s_3}\right)\right\rceil\right)}{2\left\lceil \log\left(\frac{s}{s_3}\right)\right\rceil - 1} \leq \frac{1}{\log\left(\frac{s}{s_3}\right)} \log\left(2\log\left(\frac{s}{s_3}\right) + 2\right),$$

the statement of the proposition follows easily. □

Lemma 8.6. *Let $s > s_3$, L as in Lemma 8.5 and $V \subseteq [0, L-1]$. Then the set $Z_{s,L}(V)$ is covered by at most*

$$c_Z e^{c(s)L + h_m(T)(L - \frac{1}{2}|V|)}$$

L-boxes, where $c(s) \to 0$ as $s \to \infty$. The constant c_Z does not depend on s, L and V.

Proof. The proof is similar to that of Lemma 7.3 with $s = s'$. Without loss of generality we may assume that $\mathcal{W} \subseteq \mathcal{X}_{\leq s_3}$ and we choose a "radius" r for the box D_r below the injectivity radius of $\mathcal{W}$. We can cover $Z_{s,L}(V)$ with finitely many 1-boxes xD_r with $x \in Z_{s,L}(V)$, say

$$Z_{s,L}(V) \subseteq \bigcup_{j=1}^{c_{\mathcal{W}}} x_j D_r. \tag{19}$$

The necessary number $c_{\mathcal{W}}$ of such 1-boxes is bounded by a constant independent of s, L and V. Suppose now that $x_0 D_r$ is one of the sets used in the covering (19) and consider

$$\mathcal{Z} := Z_{s,L}(V) \cap x_0 D_r.$$

If $\mathcal{Z}$ is covered by say d_1 ℓ-boxes with center in $\mathcal{Z}$, then a subsequent excursion into $\mathcal{X}_{>s}$ of length ℓ_1 has the effect that $\mathcal{Z}$ can be covered by

$$d_1 e^{\frac{1}{2} h_m(T) \ell_1}$$

$(\ell+\ell_1)$-boxes. Whereas a stay in $\mathcal{X}_{\leq s}$ of length ℓ_2 only leads to a trivial estimate, that is $\mathcal{Z}$ can be covered by

$$d_1 e^{h_m(T)\ell_2}$$

$(\ell+\ell_2)$-boxes. In total, $\mathcal{Z}$ can be covered by

$$c^m e^{h_m(T)\left(L-\frac{1}{2}|V|\right)}$$

L-boxes, where c is a constant independent of s, L and V, and m is the number of excursions into $\mathcal{X}_{>s_3}$. The proof of Lemma 8.5 now shows that

$$m \leq \frac{2L}{\log\left(\frac{s}{s_3}\right)} + 1.$$

This completes the proof. □

Proposition 8.7. *We have*

$$\nu(\mathcal{X}) \geq 1 - \frac{\delta_0 + 4\varepsilon_0}{2h_m(T)}.$$

Proof. For $L \in \mathbb{N}$ set

$$\mu_L := \frac{1}{L}\sum_{j=0}^{L-1} T_*^j \mu.$$

Then μ_L converges to ν in the weak* topology. For any $s > 0$ we have

$$\begin{aligned}
\mu_L(\mathcal{X}_{>s}) &= \frac{1}{L}\sum_{j=0}^{L-1} \mu(T^{-j}\mathcal{X}_{>s}) \\
&= \frac{1}{L}\sum_{j=0}^{L-1} \mu(\mathcal{X}_{>s} \cap T^{-j}\mathcal{X}_{>s}) + \frac{1}{L}\sum_{j=0}^{L-1} \mu(\mathcal{X}_{\leq s} \cap T^{-j}\mathcal{X}_{>s}).
\end{aligned}$$

Since $\mathcal{W}$ is bounded, the first summand vanishes for sufficiently large s. For $x \in \mathcal{X}$ we set

$$V_x := \{j \in [0, L-1] \mid T^j x \in \mathcal{X}_{>s}\}.$$

For the second summand it follows

$$\begin{aligned}
&\frac{1}{L}\sum_{j=0}^{L-1} \mu(\mathcal{X}_{\leq s} \cap T^{-j}\mathcal{X}_{>s}) = \frac{1}{L}\sum_{n=1}^{L} n\mu(\{x \in \mathcal{X}_{\leq s} \mid |V_x| = n\}) \\
&\quad = \frac{1}{L}\sum_{n=1}^{\lceil \varrho L\rceil - 1} n\mu(\{x \in \mathcal{X}_{\leq s} \mid |V_x| = n\}) + \frac{1}{L}\sum_{n=\lceil \varrho L\rceil}^{L} n\mu(\{x \in \mathcal{X}_{\leq s} \mid |V_x| = n\}) \\
&\quad \leq \frac{1}{L}\left(\lceil \varrho L\rceil - 1\right)\mu(\mathcal{X}_{\leq s}) + \frac{1}{L}\cdot L \cdot \mu(\{x \in \mathcal{X}_{\leq s} \mid |V_x| \geq \varrho L\}) \\
&\quad = \varrho + \mu(\{x \in \mathcal{X}_{\leq s} \mid |V_x| \geq \varrho L\}).
\end{aligned}$$

Using Lemmas 8.5 and 8.6 for sufficiently large L allows us to estimate the latter term. To that end we set

$$f(s) := \frac{4\log\left(2\log\left(\frac{s}{s_3}\right) + 2\right)}{\log\left(\frac{s}{s_3}\right)}$$

and let κ be the radius of the L-boxes used in the covering in Lemma 8.6. Then

$$\begin{aligned}\mu(\{x \in \mathcal{X}_{\leq s} \mid |V_x| \geq \varrho L\}) &\leq \mu\left(\bigcup_{|V|\geq \varrho L} Z_{s,L}(V)\right)\\ &\leq c_Z e^{f(s)L} e^{c(s)L+h_m(T)(L-\frac{1}{2}\varrho L)} \cdot c\kappa^{d-\delta_0} e^{(-h_m(T)+\varepsilon_0+\frac{\delta_0}{4})(L-1)}\\ &= c' e^{(\widetilde{c}(s)+\varepsilon_0+\frac{\delta_0}{4}-\frac{1}{2}h_m(T)\varrho)L},\end{aligned}$$

where $\widetilde{c}(s) := f(s) + c(s)$ and c' is a constant. If we choose

$$\varrho = \varrho(s) > \frac{\delta_0 + 4\varepsilon_0 + 4\widetilde{c}(s)}{2h_m(T)},$$

then

$$\mu(\{x \in \mathcal{X}_{\leq s} \mid |V_x| \geq \varrho(s)L\}) \to 0 \quad \text{as } L \to \infty.$$

Therefore, we find $\varepsilon(L) \searrow 0$ as $L \to \infty$, such that for sufficiently large s and L we have

$$\mu_L(\mathcal{X}_{>s}) \leq \varrho(s) + \varepsilon(L).$$

In turn,

$$\mu_L(\mathcal{X}_{\leq s}) \geq 1 - (\varrho(s) + \varepsilon(L)).$$

Thus,

$$\nu(\mathcal{X}_{\leq s}) \geq 1 - \varrho(s),$$

and

$$\nu(\mathcal{X}) \geq 1 - \varrho \quad \text{for all } \varrho > \frac{\delta_0 + 4\varepsilon_0}{2h_m(T)}.$$

This proves the claim. □

Our next goal is to estimate from above how many Bowen L-balls (or L-boxes) are needed at most to cover $\mathcal{W}$. Let $\mathcal{E}_1(\mathcal{X})^T$ denote the space of T-invariant ergodic probability measures on $\mathcal{X}$.

Proposition 8.8. *For each $\varepsilon > 0$, each $\delta > 0$ and each sufficiently large (depending on ε, δ) L_0 we find a subset $Y = Y_{L_0}$ of $\mathcal{X}$ such that $\nu(Y) > \nu(\mathcal{X}) - \varepsilon$ and the set Y can be covered by at most*

$$c_r e^{(h_m(T)-\delta_0+\delta)L_0}$$

Bowen L_0-balls or L_0-boxes, where c_r only depends on $\dim G$ and the radius r used for the definition of Bowen balls or boxes.

Proof. Since Bowen L_0-balls and L_0-boxes are essentially equivalent, we may restrict here to use one such kind. Let

$$\sigma := \frac{\nu}{\nu(\mathcal{X})}$$

be the normalization of the measure ν. The ergodic decomposition of σ and the fact that $\sigma(\mathcal{O}) = 0$ yield a subset $\mathcal{X}' \subseteq \mathcal{X}$ with $\sigma(\mathcal{X}') = 1$ and a measurable map $\mathcal{X}' \to \mathcal{E}_1(\mathcal{X})^T$, $x \mapsto \sigma_x$, such that

$$\sigma = \int_{\mathcal{X}'} \sigma_x \, d\sigma(x)$$

and $\sigma_x(\mathcal{X}) = 0$ for all $x \in \mathcal{X}'$. Let $\mathcal{P}$ be any countable partition of $\mathcal{X}$ with finite partition entropy $H_\sigma(\mathcal{P}) < \infty$. For any $n \in \mathbb{N}$ and any $x \in \mathcal{X}$ let $[x]_{\mathcal{P}_0^{n-1}}$ denote the partition element in $\mathcal{P}_0^{n-1}$ which contains x. Further let

$$I_\sigma(\mathcal{P}_0^{n-1})(x) := -\log \sigma\left([x]_{\mathcal{P}_0^{n-1}}\right)$$

denote the information function of the partition $\mathcal{P}_0^{n-1}$. Then the Shannon-McMillan-Breiman Theorem states that we find a subset $\mathcal{X}'' \subseteq \mathcal{X}'$ with $\sigma(\mathcal{X}'') = 1$ such that

$$\frac{1}{n} I_\sigma(\mathcal{P}_0^{n-1})(x) \to h_{\sigma_x}(T, \mathcal{P}) \quad \text{as } n \to \infty$$

for all $x \in \mathcal{X}''$ and in L^1.

Let $\varepsilon, \delta > 0$. As a consequence of the Shannon-McMillan-Breiman Theorem we find a subset $Y_1 \subseteq \mathcal{X}''$ with $\sigma(Y_1) > 1 - \varepsilon$ and $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$ and all $x \in Y_1$ we have

$$\frac{1}{N} I_\sigma\left(\mathcal{P}_0^{N-1}\right)(x) < h_{\sigma_x}(T, \mathcal{P}) + \delta.$$

By Lemma 8.3 we have

$$h_{\sigma_x}(T) \leq h_m(T) - \delta_0$$

for each $x \in \mathcal{X}'$. Thus, for any $N \geq N_1$ and $x \in Y_1$ it follows

$$-\frac{1}{N} \log \sigma\left([x]_{\mathcal{P}_0^{N-1}}\right) < h_m(T) - \delta_0 + \delta.$$

In turn,

$$\sigma\left([x]_{\mathcal{P}_0^{N-1}}\right) \geq e^{-N(h_m(T) - \delta_0 + \delta)}.$$

A slight modification of the proof of 7.53 in [EL10] (we provide the details in Section 9) allows us to choose a relatively compact subset $Q \subseteq \mathcal{X}$ such that $\sigma(Q) > 1 - \varepsilon$ and a partition $\mathcal{P}$ with finite partition entropy such that

$$[x]_{\mathcal{P}_0^{N-1}} \subseteq x B_N$$

for all $x \in Q$ and all $N \in \mathbb{N}$. The radius used here for the definition of Bowen balls is an injectivity radius of Q. Set $Y := Y_1 \cap Q$. Then $\sigma(Y) > 1 - 2\varepsilon$. Let $N \geq N_0$. Then Y can be covered by finitely many Bowen N-balls. In the following we deduce an upper bound on the number of necessary Bowen N-balls. Suppose

$$Y \subseteq \bigcup_{j=1}^{\ell_N} x_j B_N \qquad \text{with } x_j \in \mathcal{X} \text{ (pairwise disjoint)}$$

such that ℓ_N is minimal. Then

$$3^{\dim G} \geq \sum_{j=1}^{\ell_N} \sigma(x_j B_N),$$

where we used that the radius of the Bowen balls is an injectivity radius for Y. If $x_j B_N \cap Y \neq \emptyset$, then the intersection can be covered by at most $3^{\dim G}$

Bowen N-balls with center in Y. In the following we suppose that $x_j \in Y$ for all $j = 1, \ldots, \ell_N$. Then

$$\begin{aligned} 9^{\dim G} &\geq \sum_{j=1}^{\ell_N} \sigma(x_j B_N) \geq \sum_{j=1}^{\ell_N} \sigma\left([x_j]_{\mathcal{P}_0^{N-1}}\right) \\ &\geq \ell_N \cdot e^{-N(h_m(T)-\delta_0+\delta)}. \end{aligned}$$

Therefore

$$\ell_N \leq 9^{\dim G} e^{-N(h_m(T)-\delta_0+\delta)}.$$

This completes the proof. □

Lemma 8.9. *For sufficiently large $L_0 \in \mathbb{N}$, each $Y = Y_{L_0}$ (as in Proposition 8.8) and each sufficiently large $K \in \mathbb{N}$ there exists $n \in \{0, \ldots, L_0 - 1\}$ such that*

$$\frac{1}{K} \sum_{k=0}^{K-1} T_*^{kL_0} \mu(Y a_e^{-n}) > 1 - \frac{\delta_0 + 4\varepsilon_0}{2h_m(T)} - 4\varepsilon.$$

Proof. Since

$$\frac{1}{L} \sum_{j=0}^{L-1} T_*^j \mu(Y) \to \nu(Y) \quad > \nu(\mathcal{X}) - \varepsilon \qquad \text{as } L \to \infty,$$

we have

$$\frac{1}{L} \sum_{j=0}^{L-1} T_*^j \mu(Y) > \nu(\mathcal{X}) - 2\varepsilon$$

for sufficiently large L. For each such L let $K \in \mathbb{N}$ be maximal with $KL_0 \leq L$. Reordering the action into step length of L_0 we find

$$\begin{aligned} \frac{1}{L} \sum_{j=0}^{L-1} T_*^j \mu(Y) &= \frac{1}{L} \left(\sum_{n=0}^{L_0-1} \sum_{k=0}^{K-1} T_*^{kL_0+n} \mu(Y) + \sum_{j=KL_0}^{L-1} T_*^j \mu(Y) \right) \\ &= \sum_{n=0}^{L_0-1} \frac{K}{L} \cdot \frac{1}{K} \sum_{k=0}^{K-1} T_*^{kL_0+n} \mu(Y) + \frac{1}{L} \sum_{j=KL_0}^{L-1} T_*^j \mu(Y). \end{aligned}$$

The last sum has at most L_0 summands. Moreover, $K/L \leq 1/L_0$. Therefore

$$\nu(\mathcal{X}) - 2\varepsilon < \frac{1}{L} \sum_{j=0}^{L-1} T_*^j \mu(Y) \leq \frac{1}{L_0} \sum_{n}^{L_0-1} \frac{1}{K} \sum_{k=0}^{K-1} T_*^{kL_0+n} \mu(Y) + \frac{L_0}{L}.$$

For sufficiently large L it follows

$$\nu(\mathcal{X}) - 3\varepsilon < \frac{1}{L_0} \sum_{n=0}^{L_0-1} \frac{1}{K} \sum_{k=0}^{K-1} T_*^{kL_0+n} \mu(Y).$$

Therefore, for sufficiently large K we find $n \in \{0, \ldots, L_0 - 1\}$ such that

$$\frac{1}{K} \sum_{k=0}^{K-1} T_*^{kL_0+n} \mu(Y) > \nu(\mathcal{X}) - 4\varepsilon.$$

Proposition 8.7 finishes the proof. □

Proposition 8.10. *For sufficiently large $L_0, K \in \mathbb{N}$, the set $\mathcal{W}$ can be covered by at most*

$$b_3 e^{(h_m(T)+\delta)L_0} e^{b_1 \log K - b_2 K} e^{(h_m(T)+2\varepsilon_0 - \frac{1}{2}\delta_0 + \delta + \varepsilon')L_0 K}$$

L_0K-boxes with positive constants b_1, b_2, b_3 not depending on L_0 or K, and ε' linearly depending on ε.

Proof. Let $L_0, K \in \mathbb{N}$ be given and let $Y = Y_{L_0}$ be as in Proposition 8.8. We pick a compact set $\mathcal{K}$ which contains both Y and $\mathcal{W}$, and we choose the radius for the boxes to be an injectivity radius of $\mathcal{K}$. To start suppose that the number n (depending on K) in Lemma 8.9 is 0. Suppose that $\mathcal{W}$ is covered by ℓ 1-boxes with center in $\mathcal{K}$. Let $x_0 D_r$ be one of these. Further fix a covering of Y by $e^{(h_m(T)-\delta_0+\delta)L_0}$ L_0-boxes with center in $\mathcal{K}$. In the classical way of estimating how many Bowen KL_0-boxes are needed to cover $x_0 D_r$ we have to count how many of the sets

$$x_0 D_r \cap x_1 D_r a_e^{-L_0} \cap \ldots \cap x_{K-1} D_r a_e^{-(K-1)L_0} \tag{20}$$

are non-empty, where each $x_j D_r a_e^{-jL_0}$ is either an element of the covering of Y or arises from a trivial covering of $x_0 D_r a_e^{jL_0}$ by $e^{(h_m(T)+\delta)L_0}$ L_0-boxes. In any case, all x_j are in $\mathcal{K}$. A straightforward consideration shows that each set of the form (20) is contained in a single L_0K-box with radius $2r$, hence in $2^{\dim G}$ L_0K-boxes with radius r. In the following we estimate how many sets of the form (20) arise.

For each $V \subseteq [0, K-1]$ we set

$$\mathcal{W}(V) := \left\{ x \in \mathcal{W} \,\middle|\, \forall\, k \in [0, K-1]\colon \left(T^{kL_0} x \in Y \Leftrightarrow k \in V \right) \right\}.$$

Then decompose $\mathcal{W}$ into small sets, all of about the same measure, such that each of the small sets is contained in precisely one $\mathcal{W}(V)$. Since closely nearby points show the same behavior with respect to whether or not being contained in Y over limited times (here: K), this is indeed possible. Set

$$b := \frac{\delta_0 + 4\varepsilon_0}{2h_m(T)} + 4\varepsilon.$$

By Lemma 8.9, on average in each step $(1-b)$ of the small sets are contained in Y and b are not contained in Y. Thus, up to boundary considerations, for fixed x_0 we have

$$\left(e^{(h_m(T)-\delta_0+\delta)L_0}\right)^{(1-b)K} \left(e^{(h_m(T)+\delta)L_0}\right)^{bK} \binom{K}{bK} \tag{21}$$

sets of the form (20). There might arise counting mistakes if $x_0 D_r a^{jL_0}$ overlaps the boundary of Y. To compensate for these, we multiply (21) with $2^{\dim G}$, the number of L_0K-boxes at most necessary to cover a given L_0K-box and its boundary.

To take care of the case $n \neq 0$, we start with of most L_0 trivial estimates. Thus in total we need at most

$$\ell 2^{\dim G} e^{(h_m(T)+\delta)L_0} \left(e^{(h_m(T)-\delta_0+\delta)L_0}\right)^{(1-b)K} \left(e^{(h_m(T)+\delta)L_0}\right)^{bK} \binom{K}{bk}$$

L_0K-boxes to cover $\mathcal{W}$.

We note that

$$\delta_0 b = \delta_0 \left(\frac{\delta_0 + 4\varepsilon_0}{2h_m(T)} + 4\varepsilon \right) \leq \frac{1}{2}\delta_0 + 2\varepsilon_0 + \varepsilon'$$

with $\varepsilon' := 4\delta_0\varepsilon$. Moreover, by Stirling's formula we have

$$\begin{aligned}\binom{K}{bK} &= \frac{K!}{(bk)!((1-b)K)!} \\ &\sim \frac{\sqrt{2\pi K}\frac{K^K}{e^K}}{\sqrt{2\pi bK}\frac{(bK)^{bK}}{e^{bK}}\sqrt{2\pi(1-b)K}\frac{((1-b)K)^{(1-b)K}}{e^{(1-b)K}}} \\ &= (2\pi b(1-b)K)^{-\frac{1}{2}}\, b^{-bK}(1-b)^{(b-1)K}.\end{aligned}$$

Thus, for appropriate choices of $b_1, b_2 > 0$, it follows that

$$\binom{K}{bK} \leq e^{b_1 \log K - b_2 K}$$

for sufficiently large K. This proves the claim. □

Proof of Theorem 8.2. Combining Proposition 8.10 and Lemma 8.4 yields

$$\mu(\mathcal{W}) \leq e^{b_1 \log K - b_2 K} cK^{d - \frac{\delta_0}{4}} e^{(3\varepsilon_0 + \delta + \varepsilon' - \frac{1}{4}\delta_0)L_0 K}.$$

We choose ε so small and L_0 so large such that $\delta + \varepsilon' < \frac{1}{8}\delta$. Further we choose $\varepsilon_0 < \frac{1}{24}\delta_0$. Then the exponent $3\varepsilon_0 + \delta + \varepsilon' - \frac{1}{4}\delta_0$ is negative. Now letting K tend to infinity yields $\mu(\mathcal{W}) = 0$, which is a contradiction. □

9. Adaption of the partition from [EL10]

The partition $\mathcal{P}$ is essentially identical with the partition in [EL10, 7.51]. However our situation is slightly different to the one in [EL10] for which reason we outline the necessary steps of proof. The differences are as follows:

- The measure σ is not necessarily ergodic, and we cannot reduce to an ergodic situation as in [EL10].
- We want to find a big set Q on which the inclusion relation

$$[x]_{\mathcal{P}_0^{N-1}} \subseteq xB_N$$

  holds for all $N \in \mathbb{N}$ (in [EL10], the mass of Q does not matter as long as it is positive, and it is asked for the (weaker) relation

$$[x]_{\mathcal{P}_0^\infty} \subseteq xB_N$$

  for $N \in \mathbb{N}$ such that $xa^N \in Q$).
- We do not need the lower bounds on the atoms, which allows to simplify $\mathcal{P}$ a bit.

The construction of $\mathcal{P}$ and the proofs of its properties proceeds in a number of steps.

1) Pick a subset $Q \subseteq \mathcal{X}$ which is open, relatively compact and which has mass $\sigma(Q) > 1 - \varepsilon$. Pick an injectivity radius r of Q and decompose Q (up to measure zero) into finitely many subsets $Q_1, \ldots, Q_R$ with positive measure

such that each of these subsets is contained in a box (or Bowen ball) with radius $r/16$. Set

$$\mathcal{Q} := \{Q_1, \dots, Q_R, \mathcal{X} \setminus Q\}.$$

2) For each $i = 1, \dots, R$ and each $j \in \mathbb{N}$ let Q_{ij} be the set of points $x \in Q_i$ which return to Q with the j-th step but not earlier, that is

$$Q_{ij} = \{x \in Q_i \mid xa_e^j \in Q,\ xa_e^\ell \notin Q \text{ for } \ell = 1, \dots, j-1\}.$$

Set

$$\widetilde{\mathcal{Q}} := \{\mathcal{X} \setminus Q,\ Q_{ij} \mid i = 1, \dots, R,\ j \in \mathbb{N}\}.$$

3) We now decompose the partition elements Q_{ij} into smaller subsets. For this we remark that each box $xD_{r/16}$ is covered by at most

$$ce^{\kappa j}$$

boxes with radius $e^{-j}r/8$ (here $c \leq 2^{\dim G}$ and $\kappa \leq \dim G$). Let $i \in \{1, \dots, R\}$ and $j \in \mathbb{N}$. Note that $Q_{ij} \subseteq x_i D_{r/16}$ for some $x_i \in Q$. We fix a cover $B_1, \dots B_{N(j)}$ with $N(j) \leq ce^{\kappa j}$ by boxes with radius $e^{-j}r/8$. We define

$$\begin{aligned} Q_{ij1} &:= Q_{ij} \cap B_1 \\ Q_{ij2} &:= Q_{ij} \cap \big(B_2 \setminus B_1\big) \\ Q_{ij3} &:= Q_{ij} \cap \big(B_3 \setminus \big(B_1 \cup B_2\big)\big) \end{aligned}$$

and so on. Set

$$\mathcal{P} := \{Q_{ijk},\ \mathcal{X} \setminus Q \mid i = 1, \dots, R,\ j \in \mathbb{N},\ k = 1, \dots, N(j)\}.$$

Lemma 9.1. *For σ-a.e. $x \in Q$ we have*

$$[x]_{\mathcal{P}_0^{N-1}} \subseteq xB_N$$

for all $N \in \mathbb{N}$.

Proof. By the Poincaré Recurrence Theorem, σ-a.e. point in Q returns infinitely often to Q. We restrict to these point and pick such an x. Let $N \in \mathbb{N}$ and let

$$y \in [x]_{\mathcal{P}_0^{N-1}}.$$

There exist $i \in \{1, \dots, R\}$, $j \in \mathbb{N}$ and $k \in \{1, \dots, N(j)\}$ such that $x, y \in Q_{ijk}$. Thus, there exists $g \in D_{e^{-j}r/4}$ such that $y = xg$. Then for any $\ell = 0, \dots, j$ we have

$$g \in a_e^\ell D_{e^{\ell-j}r/4} a_e^{-\ell} \subseteq a_e^\ell B_r^G a_e^{-\ell}.$$

In the case $j \geq N-1$ it follows immediately

$$y \in xB_N.$$

Suppose that $j < N-1$. Then we find i_2, j_2, k_2 such that $xa_e^j, ya_e^j \in Q_{i_2 j_2 k_2}$. Since

$$ya_e^j = xa_e^j\big(a_e^{-j} g a_e^j\big),$$

and $a_e^{-j} g a_e^j \in B_r^G$ and r is an injectivity radius, it follows that actually

$$a_e^{-j} g a_e^j \in D_{e^{-j_2}r/4}.$$

Now reasoning inductively as above finally shows $y \in xB_N$. □

Lemma 9.2. *The partition $\mathcal{P}$ has finite partition entropy $H_\sigma(\mathcal{P})$.*

Proof. The only difference to the proof in [EL10] is that $\mathcal{X} \setminus Q$ need not be a null set. However, it mass is bounded by ε. In turn, we have

$$\sum_{i,j} j\sigma(Q_{ij}) \leq 1.$$

Taking these differences into account the proof is parallel to that in [EL10]. □

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